MAXIMAL 2-EXTENSIONS OF PYTHAGOREAN FIELDS AND RIGHT ANGLED ARTIN GROUPS

by

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Abstract. — In this paper, we describe minimal presentations of maximal pro-2 quotients of absolute Galois groups of formally real Pythagorean fields of finite type. For this purpose, we introduce a new class of pro-2 groups: Δ-Right Angled Artin groups.

We show that maximal pro-2 quotients of absolute Galois groups of formally real Pythagorean fields of finite type are Δ-Right Angled Artin groups. Conversely, let us assume that a maximal pro-2 quotient of an absolute Galois group is a Δ-Right Angled Artin group. We then show that the underlying field must be Pythagorean, formally real and of finite type. As an application, we provide an example of a pro-2 group which is not a maximal pro-2 quotient of an absolute Galois group, although it has Koszul cohomology and satisfies both the Kernel Unipotent and the strong Massey Vanishing properties.

We combine tools from group theory, filtrations and associated Lie algebras, profinite version of the Kurosh Theorem on subgroups of free products of groups, as well as several new techniques developed in this work.

A central open problem in Galois theory and number theory is to classify profinite groups that arise as absolute Galois groups. One natural approach is to study properties that such groups must satisfy. On the one hand, Artin–Schreier [1] showed that the only non-trivial finite subgroups of absolute Galois groups are of order 2. On the other hand, the well known Bloch-Kato Conjecture imposes strong restrictions on Galois cohomology. This conjecture was proved in 2011 by Rost and Voevodsky (see [9, 47]). For some other results concerning absolute Galois groups of some special fields and some related topics in field arithmetic, we refer the interested reader to [3, 6, 7, 12].

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Positselski [41] proposed a strengthened version of the Bloch–Kato Conjecture. Let us fix a prime p. A profinite group is said to satisfy the Koszul property (at p) if its cohomology algebra over $\mathbb{F}_p$ (with trivial action) is Koszul. This means that the cohomology algebra is generated in degree 1, with relations of degree 2, and admits a linear resolution. Positselski conjectured that absolute Galois groups, of fields containing a p-th root of unity, satisfy the Koszul property. This property has significant applications in current algebra and geometry. We direct the reader to [40] for a comprehensive exposition.

In [37, 38], the last two authors proposed further conjectures describing properties on absolute Galois groups: the Massey Vanishing and the Kernel Unipotent properties. The Massey Vanishing Conjecture has attracted considerable attention. Harpaz and Wittenberg [13] proved it for all algebraic number fields. For some recent significant results on this conjecture and related problems in Galois theory, we refer to the work of Merkurjev and Scavia in [24–28]. We direct the reader to their lecture notes [23] for a complete exposition on the Massey Vanishing Conjecture.

This paper focuses on the class $\mathscr{P}$ of maximal pro-2 quotients of absolute Galois groups of formally real Pythagorean fields of finite type (RPF fields). Observe that these groups are finitely generated as pro-2 groups. The class $\mathscr{P}$ is deeply connected with Witt rings, orderings of fields, and the Milnor Conjecture. We refer to the work of the third author and Spira [29, 31, 35, 36], Marshall [22], Jacob [14], Efrat-Haran [6] and Lam [17]. The third author in [29, 30] described the class $\mathscr{P}$. It is the minimal class of pro-2 groups containing $\Delta \coloneqq \mathbb{Z}/2\mathbb{Z}$, stable under coproducts and some semi-direct products. With Spira [35, 36], they also showed that groups in $\mathscr{P}$ are characterized by finite quotients. Precisely their third Zassenhaus quotients. These results led to several consequences. A first application was given by the last two authors with Pasini and Quadrelli. They showed that $\mathscr{P}$ satisfies the Koszul property [33]. A second application was given by Quadrelli [44, Theorem 1.2]. He proved that $\mathscr{P}$ satisfies the Massey Vanishing property.

Building on these results, this paper aims to study presentations of groups in $\mathscr{P}$. As an application, we give several new examples of groups which are not pro-2 maximal quotients of absolute Galois groups. Our approach, inspired by the work of the first author [10, 11], relates $\mathscr{P}$ to the well-known class of pro-p Right Angled Artin Groups (RAAGs). See [2] for a general introduction on RAAGs. This class has recently played an important role in Galois theory. We refer to the enlightening works of Snopce and Zalesski [45, Theorem 1.2] and of Blumer, Quadrelli, and Weigel [4, Theorem 1.1].

The class of Δ-RAAGs. — Let us denote by x_0 the generator of the multiplicative group $\Delta \coloneqq \mathbb{Z}/2\mathbb{Z}$. Intuitively, a Δ-Right Angled Artin group (Δ-RAAG) is a semi-direct product of a (pro-2) Right Angled Artin Group (RAAG) by Δ. The action inverts a "natural" set of generators up to conjugacy.

Let $\Gamma \coloneqq (\mathbf{X}, \mathbf{E})$ be an undirected graph with d_Γ vertices $\{1, \dots, d_\Gamma\}$ and r_Γ edges. We recall that a pro-2 Right Angled Artin Group G_Γ is defined by a pro-2 presentation with d_Γ generators $\{x_1, \dots, x_{d_\Gamma}\}$ and relations $\{[x_i, x_j] \coloneqq x_i^{-1}x_j^{-1}x_ix_j\}_{\{i,j\}\in\mathbf{E}}$. Fix $\mathbf{z} \coloneqq (z_i)_{i=1}^{d_\Gamma}$, a d_Γ-uplet in G_Γ. We assume that there exists an action $\delta_{\mathbf{z}}\colon \Delta \to \operatorname{Aut}(G_\Gamma)$, which is well-defined and satisfies the condition:

$$\text{(conj)} \qquad \delta_{\mathbf{z}}(x_0)(x_i) \coloneqq (x_i^{-1})^{z_i} \coloneqq z_i^{-1}x_i^{-1}z_i = [z_i, x_i]x_i^{-1}, \quad \text{for } 1 \leqslant i \leqslant d_\Gamma.$$

Set $G_{\mathbf{z}} \coloneqq G_\Gamma \rtimes_{\delta_{\mathbf{z}}} \Delta$. We define the class of Δ-RAAGs as the class of all pro-2 groups given by $G_{\mathbf{z}}$ where Γ varies along all graphs. As a consequence, the presentation of the pro-2 group $G_{\mathbf{z}}$ is given by $d_\Gamma + 1$ generators and $r_\Gamma + d_\Gamma + 1$ relations:

$$(\mathbf{z}\text{-Pres})\quad G_{\mathbf{z}} \coloneqq \langle x_0, x_1, \ldots, x_{d_\Gamma} \mid \quad [x_u, x_v] = 1, [x_0, x_i^{-1}]x_i^2[x_i, z_i] = 1, x_0^2 = 1, \\ \text{for } \{u, v\} \in \mathbf{E}, 1 \leqslant i \leqslant d_\Gamma \rangle.$$

We have explicit examples from the first author [10, Proposition 3.16]: every graph Γ and family $\mathbf{z}_0 \coloneqq (1, \ldots, 1)$ gives a well-defined group $G_{\mathbf{z}_0} \coloneqq G_\Gamma \rtimes_{\delta_{\mathbf{z}_0}} \Delta$. In that case, the condition (conj) is defined by:

$$\delta_{\mathbf{z}_0}(x_0)(x_i) \coloneqq x_i^{-1}, \quad \text{for } 1 \leqslant i \leqslant d_\Gamma.$$

A major innovation of this paper is the introduction of the action $\delta_{\mathbf{z}}$, in the definition of Δ-RAAGs. Using the action $\delta_{\mathbf{z}}$ and a profinite version of the Kurosh Subgroup Theorem, we show that the class of Δ-RAAGs is stable under coproducts. This is Theorem 2.13. From Proposition 2.11, we also observe that this class is stable under some specific semi-direct products. As a consequence, we conclude from [29] that $\mathscr{P}$ is a subclass of Δ-RAAGs. The study of Δ-RAAGs is also motivated by Proposition 1.3, which allows us, from techniques used by the first author [10, 11], to recover the Zassenhaus filtration of a Δ-RAAG from its underlying graph.

Our results. — Let K be a field of characteristic different from 2. Denote by G_K the maximal pro-2 quotient of its absolute Galois group. Define $L \coloneqq K(\sqrt{-1})$. We assume that K is a formally real Pythagorean field of finite type (abbreviated RPF field). This means that (*i*) -1 is not a square, (*ii*) the sum of two squares is a square, (*iii*) the group $K^\times/K^{\times 2}$ is finite.

Observe that $\Delta \simeq \mathrm{Gal}(L/K)$ and we have an exact sequence of pro-2 groups:

$$(1)\qquad 1 \to G_L \to G_K \to \Delta \to 1.$$

The following result is analogous to [45, Theorem 1.2] and [4, Theorem 1.1], and also relies on graph theory. It gives a new property satisfied by pro-2 maximal quotients of absolute Galois groups.

***Theorem A* (Theorems 2.1 and 3.4).** — *If K is a RPF field, then the exact sequence* (1) *splits. The pro-2-group G_L is RAAG, and G_K is Δ-RAAG. Conversely, let K be a field, and assume that G_K is a Δ-RAAG. Then K is a RPF field. Moreover, G_K is uniquely determined by its underlying graph.*

From presentations of pro-2 groups given by Theorem A, we get new examples of pro-2 groups in $\mathscr{P}$ (Example 2.17). We also infer new examples of pro-2 groups which are not maximal pro-2 quotients of absolute Galois groups (Example 3.5). Furthermore, if K is a RPF field, Theorem A shows that the group G_K admits a quadratic presentation. From sections 2 and 3, this presentation is given by ($\mathbf{z}$-Pres). This allows us to positively answer a question raised by Weigel [48] for groups in $\mathscr{P}$. See Proposition 3.6. From [11, Proposition 1] and [19], this question is related to the Positselski Conjecture. Let us highlight the following consequence from the Δ-RAAG theory.

***Corollary* (Proposition 3.6).** — *Assume that G is in $\mathscr{P}$. Then the presentation* ($\mathbf{z}$-Pres) *of G, given by Theorem A is minimal. Let us denote by Γ the underlying graph of G. Then:*

$$H^\bullet(G, t) \coloneqq \sum_n \dim_{\mathbb{F}_2} H^n(G)t^n = \frac{\Gamma(t)}{1-t}.$$

Here $\Gamma(t) \coloneqq \sum_n c_n(\Gamma)t^n$, with $c_n(\Gamma)$ the number of n-cliques of Γ, i.e. maximal complete subgraphs of Γ with n vertices.

Let us recall that d_Γ and r_Γ are the number of vertices and edges of Γ. The previous corollary tells us that the minimal number of generators and relations of G is $d_\Gamma + 1$ and $r_\Gamma + d_\Gamma + 1$.

We conclude this paper with the first known example of a pro-2 group which satisfies the Koszul, the (strong) Massey Vanishing and the Kernel Unipotent properties, but is not a maximal pro-2 quotient of an absolute Galois group. This example is studied in detail in Section 4.

***Theorem B* (Theorem 4.1).** — *The pro-2 group*

$$G \coloneqq \langle x_0, x_1, x_2, x_3, x_4 | \quad [x_1, x_2] = [x_2, x_3] = [x_3, x_4] = [x_4, x_1] = 1, \\ x_0^2 = 1, \quad x_0 x_j x_0 x_j = 1, \forall j \in [\![1; 4]\!]\rangle$$

does not occur as a maximal pro-2 quotient of an absolute Galois group. But it satisfies the Koszul, the (strong) Massey Vanishing and the Kernel Unipotent properties.

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List of notations

This table summarizes the main symbols and notations used in the paper. Most of these notations (and related techniques) were used by the first author in [10, 11].

Group and Galois theoretic notations. — We start with some algebraic notations. All subgroups of a profinite group will be considered closed.

Symbol / Notation	**Description**
K, L	K a field of characteristic different from 2, $L \coloneqq K(\sqrt{-1})$.
G_K, G_L	The maximal pro-2 quotient of the absolute Galois group of K, L.
RPF fields	Formally real Pythagorean fields of finite type.
$\mathscr{P}$	The class of maximal pro-2 quotients of absolute Galois groups of RPF fields.
$\Delta \coloneqq \mathbb{Z}/2\mathbb{Z}$	Multiplicative group with two elements, generated by x_0.
$F(d), F$	Free pro-2 group on d generators; written as F when d is clear.
R	Normal closed subgroup of F generated by the relations $l_1, \dots, l_r$.
$G \coloneqq F/R$	A finitely presented pro-2 group presented by generators $\{x_0, x_1, \dots, x_d\}$ and relations $\{l_1, \dots, l_r\}$.
$x^y \coloneqq y^{-1}xy$	Conjugation of x by y.
$[x, y] \coloneqq x^{-1}y^{-1}xy$	Commutator of x and y.
$[H, H]$	Normal closure of the subgroup generated by commutators of elements in H, a subgroup of G.
H^2	Normal closure of the subgroup generated by all squares in H, a subgroup of G.
H_1H_2	Normal closure of the subgroup generated by ab, for $a \in H_1$, $b \in H_2$ and H_1 and H_2 subgroups of G.
$\mathbb{U}_n$	Group of $n \times n$ upper triagular unipotent matrices over $\mathbb{F}_2$.
$\mathbb{I}_n$	Identity matrix of size $n \times n$ over $\mathbb{F}_2$.
$H^n(G)$	The n-th continuous cohomology group of the trivial G-module $\mathbb{F}_2$.
$H^\bullet(G) \coloneqq \bigoplus_{n\in\mathbb{N}} H^n(G)$	Graded cohomology ring.
$H^\bullet(G, t) \coloneqq \sum_n \dim_{\mathbb{F}_2} H^n(G)t^n$	Poincaré series of G. It is defined when $\dim_{\mathbb{F}_2} H^n(G)$ is finite for every n in $\mathbb{N}$.
$E(G)$	Completed group algebra of G over $\mathbb{F}_2$.
$E_n(G)$	The n-th power of the augmentation ideal of $E(G)$, for n in $\mathbb{N}$.
$G_n \coloneqq \{g \in G \mid g - 1 \in E_n(G)\}$	Zassenhaus filtration of G.

Lie algebras associated to filtrations. — We continue with some notations on Lie algebras coming from the Zassenhaus filtrations:

Symbol / Notation	Description
$\mathscr{L}(G) \coloneqq \bigoplus_{n\in\mathbb{N}} \mathscr{L}_n(G)$	Graded Lie algebra where $\mathscr{L}_n(G) \coloneqq G_n/G_{n+1}$.
$\mathscr{E}(G) \coloneqq \bigoplus_{n\in\mathbb{N}} \mathscr{E}_n(G)$	Graded algebra where $\mathscr{E}_n(G) \coloneqq E_n(G)/E_{n+1}(G)$.
$gocha(G,t) \coloneqq \sum_{n\in\mathbb{N}} c_n(G)t^n$	Gocha series of G, where $c_n \coloneqq \dim_{\mathbb{F}_2} E_n(G)/E_{n+1}(G)$. This name comes from Golod and Shafarevich.
$\mathscr{L}$, E, $\mathscr{E}$	Restricted free graded Lie algebra, algebra of noncommutative polynomials, and noncommutative series algebra on $\{X_0,\dots,X_d\}$ over $\mathbb{F}_2$, where each X_i has degree 1.
$\psi\colon E(F_{d+1}) \to E$	Magnus isomorphism sending $x_i \mapsto 1 + X_i$.
l, n_l	We denote by l an element in F and n_l is the least integer n such that l is in $F_n\backslash F_{n+1}$.
I, $\mathscr{I}$ and $\mathscr{J}$	Ideal (closed, graded and Lie-graded) generated by $\{\psi(l)-1 \mid l\in R\}$, the image of $\psi(l)-1$ in E_{n_l}/E_{n_l+1} for l in R and the image of l in F_{n_l}/F_{n_l+1}.
$\psi_G\colon E(G) \to E/I$	It comes from the Magnus isomorphism. Observe that we have: $\mathscr{E}(G) \simeq \mathscr{E}/\mathscr{I}$ and $\mathscr{L}(G) \simeq \mathscr{L}/\mathscr{J}$.
$[\bullet,\bullet]$	Lie bracket of a Lie algebra.

Graph theoretical notations. — We finish by notations from graph theory:

Symbol / Notation	Description
$\Gamma \coloneqq (X,E)$	An undirected graph with set of vertices set X and set of edges E.
Γ_0	The graph with one vertex.
$\Gamma_1 \simeq \Gamma_2$	We say that we have a graph isomorphism between Γ_1 and Γ_2, if we have a bijection between the vertices of Γ_1 and Γ_2 which preserves the edges.
d_Γ, r_Γ	Numbers of vertices and edges of Γ.
Γ^f, Γ^c	The graphs Γ^f and Γ^c are the free and the complete graphs on d_Γ vertices. The graph Γ^f does not have edges and the graph Γ^c has $\frac{d_\Gamma(d_\Gamma-1)}{2}$ edges.
∇	Join of two graphs.
$\coprod$	Either coproduct of pro-p groups or disjoint union of graphs.
$c_n(\Gamma)$	Number of n-cliques (maximal complete subgraphs with n vertices), for a positive integer n.
$\Gamma(t) \coloneqq \sum_{n\in\mathbb{N}} c_n(\Gamma)t^n$	Clique polynomial.
G_Γ	Pro-2 RAAG (Right Angled Artin Group) associated to the graph Γ.

$\mathscr{E}_\Gamma$, $\mathscr{L}_\Gamma$, E_Γ	Quotients of (Lie, noncommutative) algebras $\mathscr{E}$, $\mathscr{L}$ and E by the (Lie) ideal generated by $\{[X_u, X_v]\}_{\{u,v\}\in\mathbf{E}}$. We denote the gradation of $\mathscr{E}_\Gamma$ by $\{\mathscr{E}_{\Gamma,n}\}_{n\in\mathbb{N}}$ and the gradation of $\mathscr{L}_\Gamma$ by $\{\mathscr{L}_{\Gamma,n}\}_{n\in\mathbb{N}}$.
$\mathbf{z} \in G_\Gamma^{d_\Gamma}$, and $\delta_{\mathbf{z}} \colon \Delta \to \mathrm{Aut}(G_\Gamma)$	A d_Γ-tuple $\mathbf{z} \coloneqq (z_1, \ldots, z_{d_\Gamma})$, and an action $\delta_{\mathbf{z}}$ satisfying the equality (conj).
$G_{\mathbf{z}} \coloneqq G_\Gamma \rtimes_{\delta_{\mathbf{z}}} \Delta$	Δ-RAAG defined by an underlying graph Γ and the action $\delta_{\mathbf{z}}$. These groups are presented by ($\mathbf{z}$-Pres).
$\chi_0, \psi_1, \ldots, \psi_{d_\Gamma}$	χ_0 and ψ_i are the characters in $H^1(G_{\mathbf{z}})$ associated to x_0 and x_i, for $1 \leqslant i \leqslant d_\Gamma$.
$Gr(\mathscr{P})$	The class of underlying graphs coming from $\mathscr{P}$. This is justified by Theorem A. Precisely the graph Γ lies in $Gr(\mathscr{P})$ if and only if there exists a RPF field K such that $G_\Gamma = G_L$, with $L \coloneqq K(\sqrt{-1})$.

1. Introductory results on Δ-RAAGs

In this section, we study the Zassenhaus filtration on Δ-RAAGs.

1.1. Filtrations on Δ-RAAGs. — Let $G_{\mathbf{z}}$ be a Δ-RAAG. We start this section by studying the subgroup $G_{\mathbf{z},2} = G_{\mathbf{z}}^2[G_{\mathbf{z}}, G_{\mathbf{z}}]$ which is also known as the Frattini subgroup of $G_{\mathbf{z}}$.

Lemma 1.1. — *We have $[G_{\mathbf{z}}, G_{\mathbf{z}}] = G_{\mathbf{z},2} = G_{\Gamma,2} = G_\Gamma^2$.*

Proof. — Let us first notice that G_Γ is an open subgroup of $G_{\mathbf{z}}$. Thus $G_{\Gamma,2}$ is an open subgroup of $G_{\mathbf{z},2}$. From now, we need the following equalities:

$$(i)\ [x,y] = x^{-2}(xy^{-1})^2y^2, \quad \text{and } (ii)\ [x_0, x_i^n] = x_0x_i^{-n}x_0x_i^n \equiv x_i^{2n} \pmod{[G_\Gamma, G_\Gamma]},$$

where n is an integer. The identity (i) is well-known and shows that $[G,G] \subset G^2$ for every group G. From (i) and (ii), we observe that $[G_{\mathbf{z}}, G_{\mathbf{z}}] = G_{\Gamma,2} = G_\Gamma^2$, and $G_{\mathbf{z},2} = G_{\mathbf{z}}^2$.

To conclude, let us show that $G_{\mathbf{z},2} \subset G_{\Gamma,2}$. Take $x \in G_{\mathbf{z},2}$ and N an open subgroup of $G_{\mathbf{z}}$. There exists elements $u_1, \ldots, u_k$, positive integers $n_1, \ldots, n_k$ and elements x_{j_l} in $\{x_0, \ldots, x_d, x_1^{-1}, \ldots, x_d^{-1}\}$, such that:

$$xN = u_1^2 \ldots u_k^2 N, \quad \text{for } u_i = x_{j_1} \ldots x_{j_{n_i}}.$$

Consequently, $xN = \prod_i x_{j_1}^2 \ldots x_{j_{n_i}}^2 uN$ for some $u \in [G_{\mathbf{z}}, G_{\mathbf{z}}] \subset G_{\Gamma,2}$. Let us observe that $\prod_i x_{j_1}^2 \ldots x_{j_{n_i}}^2$ is in $G_{\Gamma,2}$. Then for every open subgroup N of $G_{\mathbf{z}}$, we have $x \in G_{\Gamma,2}N$. Since $G_{\Gamma,2}$ is open in $G_{\mathbf{z}}$, we conclude that x is an element in $G_{\Gamma,2}$. □

As a consequence of the three-subgroup lemma ([5, §0.3]), we have the following result:

Lemma 1.2. — *Let n be a positive integer. We have the following equality:*

$$[G_{\mathbf{z}}, G_{\Gamma,n}]G_{\Gamma,n+1} = [G_\Gamma, G_{\Gamma,n}]G_{\Gamma,n+1} = G_{\Gamma,n+1}.$$

Proof. — Let G be a pro-2 group. We introduce $\{\gamma_n(G)\}_{n\in\mathbb{N}}$ the lower central series of G. It is defined by $\gamma_1(G) \coloneqq G$ and $\gamma_n(G) \coloneqq [G, \gamma_{n-1}(G)]$, for every positive integer $n \geqslant 2$.

From an inductive argument and the three-subgroup lemma (see for instance [5, §0.3]), we observe that for every positive integer n, we have $[x_0, \gamma_n(G_\Gamma)] \leqslant G_{\Gamma,n+1}$. Thus using Lazard's formula [5, Theorem 11.2], we infer $[x_0, G_{\Gamma,n}] \subset G_{\Gamma,n+1}$. Consequently

$$[G_{\mathbf{z}}, G_{\Gamma,n}]G_{\Gamma,n+1} = [G_\Gamma, G_{\Gamma,n}]G_{\Gamma,n+1} = G_{\Gamma,n+1}.$$

□

Let us now compute the Zassenhaus filtrations of Δ-RAAGs.

Proposition 1.3. — *For every integer $n \geqslant 2$, we have: $G_{\Gamma,n} = G_{\mathbf{z},n}$.*

As a consequence, we infer that for every positive integer $n \geqslant 2$:

$$\mathscr{L}_n(G_{\mathbf{z}}) \coloneqq G_{\mathbf{z},n}/G_{\mathbf{z},n+1} \simeq \mathscr{L}_{\Gamma,n}.$$

Proof. — The proof is done by induction. Lemma 1.1 gives us $G_{\mathbf{z},2} = G_{\Gamma,2}$.

Assume $n \geqslant 2$ and let us state our induction hypothesis: $G_{\Gamma,n} = G_{\mathbf{z},n}$. Then from [5, Theorem 12.9], the induction hypothesis, and Lemma 1.2, we infer:

$$\begin{aligned} G_{\mathbf{z},n+1}G_{\Gamma,n+1} &= G^2_{\mathbf{z},\lceil \frac{n+1}{2}\rceil} \prod_{i+j=n+1} [G_{\mathbf{z},i}, G_{\mathbf{z},j}]G_{\Gamma,n+1} \\ &= G^2_{\Gamma,\lceil \frac{n+1}{2}\rceil} \prod_{i+j=n+1} [G_{\Gamma,i}, G_{\Gamma,j}]G_{\Gamma,n+1} \\ &= G_{\Gamma,n+1}. \end{aligned}$$

Therefore $G_{\mathbf{z},n+1} \leqslant G_{\Gamma,n+1}$. Furthermore, the group G_Γ is a subgroup of $G_{\mathbf{z}}$, so for every positive integer n, we have $G_{\Gamma,n} \leqslant G_{\mathbf{z},n}$. Thus, for every integer $n \geqslant 2$, $G_{\mathbf{z},n} = G_{\Gamma,n}$.

From [46, Theorem 2.4], we infer $\mathscr{L}(G_\Gamma) \simeq \mathscr{L}_\Gamma$. This allows us to conclude. □

Let us recall the following result:

Lemma 1.4. — *Let Γ_1 and Γ_2 be two graphs. The following assertions are equivalent:*

(*i*) $G_{\Gamma_1} \simeq G_{\Gamma_2}$.

(*ii*) $\mathscr{E}_{\Gamma_1} \simeq \mathscr{E}_{\Gamma_2}$.

(*iii*) $H^\bullet(\Gamma_1) \simeq H^\bullet(\Gamma_2)$.

(*iv*) $\Gamma_1 \simeq \Gamma_2$.

Proof. — Clearly, (*iv*) implies (*i*).

The first author showed in [11, Proposition 3.4] that $\mathscr{E}(G_\Gamma) \simeq \mathscr{E}_\Gamma$. Thus (*i*) implies (*ii*).

Since $\mathscr{E}_\Gamma \simeq \mathscr{E}(G_\Gamma)$, the assertion (*ii*) gives us $\mathscr{E}(G_{\Gamma_1}) \simeq \mathscr{E}(G_{\Gamma_2})$. These two algebras are Koszul. We refer to [2, Theorem 1.2]. Therefore from [11, Proposition 3.4], we infer $H^\bullet(G_{\Gamma_1}) \simeq H^\bullet(G_{\Gamma_2})$. So (*ii*) implies (*iii*).

We conclude (*iii*) implies (*iv*) using [46, Corollary 2.14]. We also refer to [15]. □

As a consequence:

Corollary 1.5. — *Let us consider two graphs Γ_1 and Γ_2, and two vectors $\mathbf{z}_1 \in \mathrm{G}_{\Gamma_1}^{d_{\Gamma_1}}$ and $\mathbf{z}_2 \in G_{\Gamma_2}^{d_{\Gamma_2}}$. If we have an isomorphism of Δ-RAAGs, $G_{\mathbf{z}_1} \simeq G_{\mathbf{z}_2}$, then $\Gamma_1 \simeq \Gamma_2$.*

Proof. — Since $G_{\mathbf{z}_1} \simeq G_{\mathbf{z}_2}$, we observe that $d_{\Gamma_1} = d_{\Gamma_2}$. In particular, we infer that:

$$G_{\Gamma_1}/G_{\Gamma_1,2} \simeq G_{\Gamma_2}/G_{\Gamma_2,2}.$$

Moreover, from Proposition 1.3, we infer for every integer $n \geqslant 2$, that $G_{\Gamma_1,n} = G_{\Gamma_2,n}$. Thus $\mathscr{E}_{\Gamma_1} \simeq \mathscr{E}(G_{\Gamma_1}) \simeq \mathscr{E}(G_{\Gamma_2}) \simeq \mathscr{E}_{\Gamma_2}$. We conclude using Lemma 1.4. □

1.2. Third Zassenhaus quotient of Δ-RAAGs. — In this section, we study the group $\mathscr{G}_{\mathbf{z}} \coloneqq G_{\mathbf{z}}/G_{\mathbf{z},3}$. When G is a pro-2 group, we denote by G^4 the normal closure of the subgroup of G generated by the 4-th powers. Observe that we have the equality $G_3 = G^4[G^2, G]$. Let us recall that the third Zassenhaus quotient G/G_3 plays a fundamental role in the study of the maximal pro-2 quotients of absolute Galois groups of RPF fields. In that context, it is called the Witt group of G. We refer to [35, 36] for further details.

Lemma 1.6. — *Let $G_{\mathbf{z}}$ be a Δ-RAAG. For every integer $n \geqslant 1$, we have $\delta_{\mathbf{z}}(x_0)(G_{\Gamma,n}) \subset G_{\Gamma,n}$. Consequently $\delta_{\mathbf{z}}$ induces an action $\overline{\delta_{\mathbf{z}}}$ of Δ on $G_\Gamma/G_{\Gamma,3}$.*

Proof. — Let us consider two elements x and z in G_Γ. Recall that [11, Proposition 1.7] and the Magnus isomorphism gives us an isomorphism $\psi_{G_\Gamma}\colon E(G_\Gamma) \to E_\Gamma$. Let us write $\psi_{G_\Gamma}(x) \coloneqq 1 + X$ and $\psi_{G_\Gamma}(z) \coloneqq 1 + Z$ in E_Γ. We infer:

$$\psi_{G_\Gamma}(x^z) = 1 + X + [X, Z] \times \frac{1}{1+Z}.$$

Thus for every positive integer n, we deduce $\delta_{\mathbf{z}}(x_0)(G_{\Gamma,n}) \subset G_{\Gamma,n}$, and so $\delta_{\mathbf{z}}$ induces an action $\overline{\delta_{\mathbf{z}}}$ on $G_\Gamma/G_{\Gamma,3}$. □

Corollary 1.7. — *We have $\mathscr{G}_{\mathbf{z}} \simeq G_\Gamma/G_{\Gamma,3} \rtimes_{\overline{\delta_{\mathbf{z}}}} \Delta$.*

Proof. — From Proposition 1.3, we infer:

$$G_\Gamma \cap G_{\mathbf{z},3} = G_\Gamma \cap G_{\Gamma,3} = G_{\Gamma,3}.$$

Consequently, we obtain an exact sequence:

$$1 \to G_\Gamma/G_{\Gamma,3} \to \mathscr{G}_{\mathbf{z}} \to \Delta \to 1.$$

Furthermore, x_0 is not in $G_{\mathbf{z},3}$. Thus the previous exact sequence splits and the action of Δ on $G_\Gamma/G_{\Gamma,3}$ is given by $\overline{\delta_{\mathbf{z}}}$. □

Corollary 1.8. — *We have $\mathscr{G}_{\mathbf{z}}^2 = [\mathscr{G}_{\mathbf{z}}, \mathscr{G}_{\mathbf{z}}]$.*

Proof. — Since $\mathscr{G}_{\mathbf{z}}$ is a 2-group, we have the inclusion $[\mathscr{G}_{\mathbf{z}}, \mathscr{G}_{\mathbf{z}}] \subset \mathscr{G}_{\mathbf{z}}^2$. We also have:

$$\begin{aligned}[\mathscr{G}_{\mathbf{z}}, \mathscr{G}_{\mathbf{z}}] &= [G_{\mathbf{z}}/G_{\mathbf{z},3}, G_{\mathbf{z}}/G_{\mathbf{z},3}] \\ &= [G_{\mathbf{z}}, G_{\mathbf{z}}]G_{\mathbf{z},3}/G_{\mathbf{z},3} \\ &= G_{\mathbf{z}}^2 G_{\mathbf{z},3}/G_{\mathbf{z},3} \\ &= \mathscr{G}_{\mathbf{z}}^2.\end{aligned}$$

□

Remark 1.9. — Similar results to the previous parts were already known for some groups in $\mathscr{P}$. We refer to [34, Corollary 4.8 and Lemma 4.12].

1.3. Gocha series and associated graded algebras. — Now, we aim to compute the algebras $\mathscr{E}(G_{\mathbf{z}})$ and $\mathscr{L}(G_{\mathbf{z}})$ as a quotient of $\mathscr{E}$ and $\mathscr{L}$. The main tools are the Magnus isomorphism and restricted Lie algebras techniques. We refer to [18, Chapitre II, Partie 3] and [5, §12.1] for further detail. We first observe that $\mathscr{E}(G)$ is the universal envelope of $\mathscr{L}(G)$. From the Magnus isomorphism, we also have the following chain of morphisms of $\mathbb{F}_2$-vector spaces $G_\Gamma/G_{\Gamma,2} \hookrightarrow F(d_\Gamma+1)/F(d_\Gamma+1)_2 \simeq E/E_2 \hookrightarrow \mathscr{L} \hookrightarrow \mathscr{E}$.

We define $\mathscr{I}_{\mathbf{z}}$ (resp. $\mathscr{J}_{\mathbf{z}}$) the two-sided ideal of $\mathscr{E}$ (resp. $\mathscr{L}$) generated by:

$$\{X_0^2, \quad [X_u, X_v], \quad \text{and} \quad [X_0, X_k] + [X_k, \epsilon_k] + X_k^2\}_{\{u,v\}\in\mathbf{E} \text{ and } 1\leqslant k\leqslant d},$$

where ϵ_k is the image of z_k in G_Γ through the previous chain of maps. Let us observe that $\epsilon_k = 0$ if and only if z_k has degree n_{z_k} larger than or equal two. Moreover, since z_k is in G_Γ, then ϵ_k can be seen as an element in $\mathscr{L}_\Gamma$. We introduce

$$\mathscr{E}_{\mathbf{z}} \coloneqq \mathscr{E}/\mathscr{I}_{\mathbf{z}}, \quad \text{and} \quad \mathscr{L}_{\mathbf{z}} \coloneqq \mathscr{L}/\mathscr{J}_{\mathbf{z}}.$$

Theorem 1.10. — *For every Δ-RAAG $G_{\mathbf{z}}$, we have the following isomorphisms of graded-$\mathbb{F}_2$ Lie algebras:*

$$\mathscr{E}(G_{\mathbf{z}}) \simeq \mathscr{E}_{\mathbf{z}}, \quad \textit{and} \quad \mathscr{L}(G_{\mathbf{z}}) \simeq \mathscr{L}_{\mathbf{z}}.$$

Proof. — As vector spaces, we have from Proposition 1.3:

$$\mathscr{L}_1(G_{\mathbf{z}}) \simeq \mathbb{F}_2 \bigoplus \mathscr{L}_1(G_\Gamma), \quad \text{and for } n \geqslant 2, \quad \mathscr{L}_n(G_{\mathbf{z}}) \simeq \mathscr{L}_n(G_\Gamma).$$

From the relations satisfied by restricted Lie algebras (see for instance [5, Equation (7) in §12.1]) and relations $X_0^2 = 0$ and $[X_0, X_k] + [X_k, \epsilon_k] + X_k^2 = 0$; we observe for every $n \geqslant 2$ that $\mathscr{L}_{\mathbf{z},n} \subset \mathscr{L}_{\Gamma,n}$. Thus $\dim_{\mathbb{F}_2} \mathscr{L}_{\mathbf{z},n} \leqslant \dim_{\mathbb{F}_2} \mathscr{L}_{\Gamma,n}$.

Furthermore we have graded surjections $\mathscr{E}_{\mathbf{z}} \to \mathscr{E}(G_{\mathbf{z}})$ and $\mathscr{L}_{\mathbf{z}} \to \mathscr{L}(G_{\mathbf{z}})$. Thus $\dim_{\mathbb{F}_2} \mathscr{L}_n(G_{\mathbf{z}}) \leqslant \dim_{\mathbb{F}_2} \mathscr{L}_{\mathbf{z},n}$. Consequently, from Proposition 1.3 and the previous discussion, we deduce for every $n \geqslant 2$ that:

$$\dim_{\mathbb{F}_2} \mathscr{L}_n(G_{\mathbf{z}}) \leqslant \dim_{\mathbb{F}_2} \mathscr{L}_{\mathbf{z},n} \leqslant \dim_{\mathbb{F}_2} \mathscr{L}_{\Gamma,n} = \dim_{\mathbb{F}_2} \mathscr{L}_n(G_{\mathbf{z}}).$$

This implies that $\mathscr{L}_{\mathbf{z}} \simeq \mathscr{L}(G_{\mathbf{z}})$. Since $\mathscr{E}_{\mathbf{z}}$ is the universal envelope of $\mathscr{L}_{\mathbf{z}}$ and $\mathscr{E}(G_{\mathbf{z}})$ is the universal envelope of $\mathscr{L}(G_{\mathbf{z}})$, we infer $\mathscr{E}_{\mathbf{z}} \simeq \mathscr{E}(G_{\mathbf{z}})$. □

Remark 1.11. — Following the terminology of [33, Definition 7.7], Theorem 1.10 shows that the presentation ($\mathbf{z}$-Pres) is quadratically defined.

Corollary 1.12. — *We have*

$$gocha(G_{\mathbf{z}}, t) = \frac{1+t}{\Gamma(-t)}.$$

Proof. — Proposition 1.3 gives us $\mathscr{L}_1(G_{\mathbf{z}}) \simeq \mathbb{F}_2 \bigoplus \mathscr{L}_{\Gamma,1}$, and for n$\geqslant 2$, $\mathscr{L}_n(G_{\mathbf{z}}) \simeq \mathscr{L}_{\Gamma,n}$. Then, from Jennings-Lazard formula [18, Proposition 3.10, Appendice A], we obtain:

$$gocha(G_{\mathbf{z}}, t) = (1+t) \times gocha(G_\Gamma, t) = \frac{1+t}{\Gamma(-t)}.$$

□

1.4. Example: $\delta_{\mathbf{z}_0}$-action. — We take $\mathbf{z}_0 \coloneqq (1,\dots,1)$ in $G_\Gamma^{d_\Gamma}$. From (conj), the action $\delta_{\mathbf{z}_0}\colon \Delta \to \mathrm{Aut}(G_\Gamma)$ is defined by $\delta_{\mathbf{z}_0}(x_0)(x_i) \coloneqq x_i^{-1}$. By [10, Proposition 3.16], this action is well-defined. Let Γ^f be a free graph on d_Γ vertices and Γ^c be the complete graph on d_Γ vertices. We observe the following facts.

- We have an isomorphism:

$$\coprod_{i=1}^{d_\Gamma+1} \Delta \simeq F(d_\Gamma) \rtimes_{\delta_{\mathbf{z}_0}} \Delta \coloneqq G_{\Gamma^f} \rtimes_{\delta_{\mathbf{z}_0}} \Delta.$$

- Every $\mathbf{z}$ in $G_{\Gamma^f}^{d_\Gamma}$ defines a Δ-RAAG $G_{\mathbf{z}}$.
- Every $\mathbf{z}$ in $G_{\Gamma^c}^{d_\Gamma}$ defines a Δ-RAAG $G_{\mathbf{z}}$. Since G_{Γ^c} is commutative, we infer:

$$G_{\mathbf{z}} \simeq \mathbb{Z}_2^{d_\Gamma} \rtimes_{\delta_{\mathbf{z}_0}} \Delta \simeq G_{\Gamma^c} \rtimes_{\delta_{\mathbf{z}_0}} \Delta.$$

Definition 1.13 **(SAP and superpythagorean groups)**. — Let us define $\mathbf{z}_0 \coloneqq (1,\dots,1) \in G_\Gamma^{d_\Gamma}$. We say that a Δ-RAAG $G_{\mathbf{z}}$ is:
– a SAP group, if $G_{\mathbf{z}} \simeq G_{\Gamma^f} \rtimes_{\delta_{\mathbf{z}_0}} \Delta$,
– a superpythagorean group, if $G_{\mathbf{z}} \simeq G_{\Gamma^c} \rtimes_{\delta_{\mathbf{z}_0}} \Delta$.

The previous definition is motivated by Definition 2.3. As we see in the next section, these groups play a fundamental role in the study of the RPF fields. Let us recall that SAP is the acronym for Strong Approximation Property. We refer to [17, pages 264 and 265] for further details.

2. Pythagorean fields

Let K be a RPF field and $L \coloneqq K(\sqrt{-1})$. We consider G_K (resp. G_L) the maximal pro-2 quotient of the absolute Galois group of K (resp. L). We denote by $K^\times$ (resp. $K^{\times 2}$) the group of invertible elements (resp. invertible squares) of K, and $|K^\times/K^{\times 2}|$ the cardinality of the $\mathbb{F}_2$-vector space $K^\times/K^{\times 2}$. The main goal of this section is to show the following result:

Theorem 2.1. — *Let K be a field. We have the following equivalence:*
(i) there exists a Δ-RAAG $G_{\mathbf{z}}$ such that $G_{\mathbf{z}} \simeq G_K$,
(ii) the field K is RPF.
Furthermore, if K is a RPF field, then the group G_K is uniquely determined by an underlying graph Γ with d_Γ vertices and an element $\mathbf{z}$ in $G_\Gamma^{d_\Gamma}$.

We introduce several results before proving Theorem 2.1.

2.1. Semi-direct product and the class $\mathscr{P}$. — First, we show that we can write G_K as a semi-direct product of G_L by Δ. For this purpose, let us recall [29, Proposition]:

Proposition 2.2. — *There exists a unique morphism $\phi\colon G_K \to \Delta$ such that $\phi(\sigma) = x_0$ for every involution σ in G_K. Furthermore, $\ker(\phi) = G_L$.*

To conclude that G_K is a semi-direct product, it is sufficient to show that the morphism ϕ defined in Proposition 2.2 admits a section. For this purpose, we will use [34, Proposition 4.9]. Before, let us recall the following definition:

Definition 2.3 **(SAP and superpythagorean fields)**. — We say that K is

– a SAP field if G_K is a SAP group, i.e. $G_K \simeq G_{\Gamma^f} \rtimes_{\delta_{\mathbf{z}_0}} \Delta$,
– a superpythagorean field if G_K is a superpythagorean field, i.e. $G_K \simeq G_{\Gamma^c} \rtimes_{\delta_{\mathbf{z}_0}} \Delta$.

Example 2.4. — We give here a few examples of RPF fields. We refer to [17, Chapter 8, section 4] for more examples and further details.

• As a first example, we can take $K \coloneqq \mathbb{R}$. Then $\mathbb{R}$ is a RPF field and both a SAP and a superpythagorean field. Indeed, we have $G_{\mathbb{R}} \simeq G_{\varnothing} \rtimes_{\delta_{\mathbf{z}_0}} \Delta$, where $\varnothing$ is the empty graph.

• As a second example, let us take $K \coloneqq \mathbb{R}((x_1))\ldots((x_d))$, the field of iterated Laurent series over K. Then, the field K is RPF and superpythagorean, i.e. we have $G_K \simeq G_{\Gamma^c} \rtimes_{\delta_{\mathbf{z}_0}} \Delta$.

Let us recall [34, Corollaries 4.2 and 4.11]:

Proposition 2.5. — *For every integer $d_\Gamma \geqslant 0$, there exists two RPF fields N and C such that: (i) $|C^\times/C^{\times 2}| = |N^\times/N^{\times 2}| = 2^{d_\Gamma+1}$, (ii) the field N is SAP, and (iii) the field C is superpythagorean.*

Furthermore, we fix K a RPF field with $|K^\times/K^{\times 2}| = 2^{d_\Gamma+1}$. Then there exists N and C as before, such that the following commutative diagram, with exact rows, holds:

$$\begin{tikzcd}
1 \arrow[r] & F(d_\Gamma) \simeq G_{\Gamma^f} \arrow[r] \arrow[d, two heads] & G_N \arrow[r, "\phi_N"] \arrow[d, two heads, "\pi_N"] & \Delta \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\
1 \arrow[r] & G_L \arrow[r] \arrow[d, two heads] & G_K \arrow[r, "\phi_K"] \arrow[d, two heads, "\pi_C"] & \Delta \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\
1 \arrow[r] & \mathbb{Z}_2^{d_\Gamma} \simeq G_{\Gamma^c} \arrow[r] & G_C \arrow[r, "\phi_C"] & \Delta \arrow[r] & 1
\end{tikzcd}$$

In addition, the columns are all epimorphisms, and the first and last rows split.

We are now able to prove the following result.

Theorem 2.6. — *There exists a unique morphism $\phi_K \colon G_K \to \Delta$ such that:*
(i) for every involution $\sigma \in G_K$, we have $\phi_K(\sigma) = x_0$,
(ii) the kernel of ϕ_K is G_L,
(iii) the morphism ϕ_K admits a section ψ_K.

Proof. — By Proposition 2.2, the morphism ϕ_K satisfies (i) and (ii). Let us show that ϕ_K satisfies (iii). The map ϕ_N introduced in Proposition 2.5 admits a section ψ_N. We define $\psi_K \coloneqq \pi_N \circ \psi_N$. From Proposition 2.5 we have $\phi_K \circ \psi_K(x_0) = \phi_K \circ \pi_N \circ \psi_N(x_0) = x_0$. Thus ψ_K is a section of ϕ_K. □

Consequently, if K is a RPF field, we can write:

$$G_K \coloneqq G_L \rtimes_{\psi_K} \Delta.$$

Corollary 2.7. — *We have $G_K/G_K^2 \simeq G_L/G_L^2 \times \Delta$.*

Proof. — Let us use the notations from Proposition 2.5. Since G_N and G_C are Δ-RAAGs and $G_K^2 \subset G_L$, we deduce from Proposition 2.5 that $G_K^2 = G_L^2$. Consequently,

$$G_K/G_K^2 \simeq (G_K/G_L) \times (G_L/G_L^2) \simeq \Delta \times G_L/G_L^2.$$

□

2.2. Structural results. — In this part, we recall structural results on $\mathscr{P}$. Let H be a pro-2 group and let ψ be a morphism of groups $\psi\colon \Delta \to \mathrm{Aut}(H)$. We consider the semi-direct product $G := H \rtimes_{\psi} \Delta$.

Definition 2.8 **(Semi-trivial action)**. — Let n be a positive integer. We use the multiplicative notation for the product of the abelian pro-2 group $\mathbb{Z}_2^n$. We define an action i_n of G on $\mathbb{Z}_2^n$, that we call semi-trivial, by:

- for every $h \in H$ and $v \in \mathbb{Z}_2^n$, $i_n(h).v := v$,
- for every $v \in \mathbb{Z}_2^n$, $i_n(x_0).v := v^{-1}$.

Remark 2.9. — Observe that:

$$\mathbb{Z}_2^n \rtimes_{i_n} G \simeq (\mathbb{Z}_2^n \times H) \rtimes_{i_n \times \psi} \Delta,$$

where the action $i_n \times \psi$ of Δ on $\mathbb{Z}_2^n \times H$ is defined by:

- $(i_n \times \psi)(x_0).h := \psi(h)$ for every $h \in H$,
- $(i_n \times \psi)(x_0).v := i_n(x_0).v = v^{-1}$ for every $v \in \mathbb{Z}_2^n$.

Let us recall [29, Theorem] (we also refer to [33, §6.2]):

Theorem 2.10. — *The class $\mathscr{P}$ is exactly the minimal class of pro-2 groups satisfying the following conditions:*

(i) the group Δ is in $\mathscr{P}$,
(ii) if $G_1, \dots, G_m$ are in $\mathscr{P}$, then $G_1 \coprod G_2 \coprod \dots \coprod G_m$ is also in $\mathscr{P}$,
(iii) if n is a positive integer and G_K is in $\mathscr{P}$, then $\mathbb{Z}_2^n \rtimes_{i_n} G_K$ is in $\mathscr{P}$.

2.3. Structure of G_L. — Theorem 2.6 allows us to write $G_K \simeq G_L \rtimes_{\psi_K} \Delta$. From Theorem 2.10, we study the structure of G_L.

2.3.1. Semi-direct products. — Let us consider the Δ-RAAG defined by $G_{\mathbf{z}} := G_\Gamma \rtimes_{\delta_{\mathbf{z}}} \Delta$. Take Γ^c the complete graph on n vertices. We show that $G := G_{\Gamma^c} \rtimes_{i_n} G_{\mathbf{z}}$ is also Δ-RAAG.

Proposition 2.11. — *The pro-2 group G is Δ-RAAG. Precisely:*

$$G \simeq G_{\Gamma \nabla \Gamma^c} \rtimes_{\delta_{\mathbf{z}'}} \Delta,$$

where ∇ denotes the join operation of graphs and $\mathbf{z}' := (\mathbf{z}, 1, \dots, 1)$ in $G_{\Gamma\nabla\Gamma^c}^{d_\Gamma + n}$

Proof. — From Remark 2.9, we infer a $(d_\Gamma + n)$-uplet $\mathbf{z}' := (\mathbf{z}, 1, \dots, 1)$ in $G_{\Gamma\nabla\Gamma^c}^{d_\Gamma+n}$ such that:

$$G \simeq \mathbb{Z}_2^n \rtimes_{i_n} (G_\Gamma \rtimes_{\delta_{\mathbf{z}}} \Delta) \simeq (G_{\Gamma^c} \times G_\Gamma) \rtimes_{\delta_{\mathbf{z}_0} \times \delta_{\mathbf{z}}} \Delta \simeq G_{\Gamma\nabla\Gamma^c} \rtimes_{\delta_{\mathbf{z}'}} \Delta.$$

The action $\delta_{\mathbf{z}'}$ is well-defined since $\mathbb{Z}_2^n$ commutes with G_Γ inside $G_{\Gamma\nabla\Gamma^c}$. □

Assume there exists a group G_{K_1} in $\mathscr{P}$, and a positive integer n such that $G_K \simeq \mathbb{Z}_2^n \rtimes_{i_n} G_{K_1}$. Let us compare G_L and G_{L_1}:

Corollary 2.12. — *We have $G_L \simeq \mathbb{Z}_2^n \times G_{L_1}$.*

Proof. — Remark 2.9 gives us:

$$G_K \simeq \mathbb{Z}_2^n \rtimes_{i_n} \left(G_{L_1} \rtimes_{\psi_{K_1}} \Delta\right) \simeq (\mathbb{Z}_2^n \times G_{L_1}) \rtimes_{\psi_{K_1} \times i_n} \Delta.$$

From Theorem 2.6, the group G_{L_1} does not have an involution, then $G_{L_1} \times \mathbb{Z}_2^n$ also does not. So by Theorem 2.6, we conclude that $G_L \simeq G_{L_1} \times \mathbb{Z}_2^n$. □

2.3.2. Coproducts. — Here we assume that there exists two fields K_1 and K_2 such that $G_K \simeq G_{K_1} \coprod G_{K_2}$. We study G_L from G_{L_1} and G_{L_2}.

First, we need to introduce a technical result, which will also be useful later. Let Γ_1 and Γ_2 be two graphs. We denote by $\{x_{1i}, \dots, x_{1d_{\Gamma_i}}\}$ a canonical system of generators of the RAAG G_{Γ_i} for $i \in \{1,2\}$. We denote by Γ_0 the graph with one vertex, and z a generator of $G_{\Gamma_0} \simeq \mathbb{Z}_2$.

Theorem 2.13. — *The class of Δ-RAAGs is stable under finite coproducts. Precisely, we have*

$$\left(G_{\Gamma_1} \rtimes_{\delta_{\mathbf{z}_1}} \langle x_{10} \rangle\right) \coprod \left(G_{\Gamma_2} \rtimes_{\delta_{\mathbf{z}_2}} \langle x_{20} \rangle\right) \simeq G_{\Gamma_1 \coprod \Gamma_2 \coprod \Gamma_0} \rtimes_{\delta_{\mathbf{z}}} \langle x_{10} \rangle.$$

Through the previous isomorphism, z is sent to $x_{10}x_{20}$. A (canonical) system of generators of the pro-2 RAAG $G_{\Gamma_1 \coprod \Gamma_2 \coprod \Gamma_0}$ is $\{x_{11}, \dots, x_{1d_{\Gamma_1}}, x_{21}, \dots, x_{2d_{\Gamma_2}}, z\}$ and the vector $\mathbf{z}$ is given by:

$$\mathbf{z} := (z_{11}, \dots, z_{1d_{\Gamma_1}}, z \times z_{21}, \dots, z \times z_{2d_{\Gamma_2}}, 1) \in (G_{\Gamma_1 \coprod \Gamma_2 \coprod \Gamma_0})^{d_{\Gamma_1}+d_{\Gamma_2}+1}.$$

The vector $\mathbf{z}$ defines an action $\delta_{\mathbf{z}} \colon \Delta \to G_\Gamma$ satisfying (conj).

Theorem 2.13 is a consequence of [39, Theorem (4.2.1)], which is a profinite version of the Kurosh subgroup Theorem:

Theorem 2.14. — *Let $G_1, \dots, G_n$ be a collection of finitely generated pro-p groups. Assume that $G = \coprod_{i=1}^n G_i$ and H is an open subgroup of G. Define $S_i := \bigcup_{j=1}^{n_i} \{s_{i,j}\}$, where $1 \leqslant i \leqslant n$ and $n_i = |S_i|$, a system of coset representatives satisfying for every $1 \leqslant i \leqslant n$:*

$$G = \bigcup_{j=1}^{n_i} G_i s_{i,j} H.$$

Then

$$H = \coprod_{i=1}^{n} \left(\coprod_{j=1}^{n_i} G_i^{s_{i,j}} \cap H \right) \coprod F(d),$$

where $F(d)$ is a free pro-p group of rank

$$d = \sum_{i=1}^{n} ([G:H] - n_i) - [G:H] + 1, \quad \textit{and} \quad G_i^{s_{i,j}} = s_{i,j} G_i s_{i,j}^{-1}.$$

Let us now prove Theorem 2.13

Proof Theorem 2.13. — We write $G := \left(G_{\Gamma_1} \rtimes_{\delta_{\mathbf{z}_1}} \langle x_{10} \rangle\right) \coprod \left(G_{\Gamma_2} \rtimes_{\delta_{\mathbf{z}_2}} \langle x_{20} \rangle\right)$, and we define a map $\phi \colon G \to \Delta$ by:

- $\phi(x) = 0$ for every $x \in G_{\Gamma_1}$,
- $\phi(y) = 0$ for every $y \in G_{\Gamma_2}$,
- $\phi(x_{10}) = x_{10}$ and $\phi(x_{20}) = x_{10}$.

We introduce $H := \ker(\phi)$. This subgroup is closed and of index 2 in G. So H is normal and open in G. We apply Theorem [39, Theorem (4.2.1)] by taking $G_1 := G_{\mathbf{z}_1}$, $G_2 := G_{\mathbf{z}_2}$ and $S_1 = S_2 := \{1\}$.

The groups G_1 and G_2 are not included in H. Thus $G = G_1 \times S_1 \times H = G_2 \times S_2 \times H$. Note that $G := G_1 \coprod G_2$. Moreover S_1 and S_2 satisfy the hypothesis of Theorem [39, Theorem (4.2.1)]. Additionally, $G_1 \cap H = G_{\Gamma_1}$ and $G_2 \cap H = G_{\Gamma_2}$. Therefore from Theorem [39, Theorem (4.2.1)], we conclude that:

$$H \simeq G_{\Gamma_1} \coprod G_{\Gamma_2} \coprod G_{\Gamma_0} \simeq G_{\Gamma_1 \coprod \Gamma_2 \coprod \Gamma_0}.$$

Through the previous isomorphism, the element z is sent to $x_{10}x_{20}$. The map $\psi\colon \Delta \to G$, which maps x_{10} to x_{10} defines a section of ϕ, and induces an action of $\Delta \coloneqq \langle x_{10} \rangle$ on H. This action is precisely defined by $\delta_{\mathbf{z}}$. □

Remark 2.15. — More generally, from [39, Theorem (4.2.1)], we deduce the following result. Let δ_1 (resp. δ_2) be an action of $\Delta \coloneqq \langle x_0 \rangle$ (resp. $\Delta \coloneqq \langle y_0 \rangle$) on a pro-2 group A (resp. B) generated by x_i (resp. y_j). Then

$$(A \rtimes_{\delta_1} \langle x_0 \rangle) \coprod (B \rtimes_{\delta_2} \langle y_0 \rangle) \simeq \left(A \coprod B \coprod G_{\Gamma_0}\right) \rtimes_{\delta_1 * \delta_2} \langle x_0 \rangle.$$

A generator z of G_{Γ_0} is sent to $x_0 y_0$ through the previous isomorphisms. The action $\delta_1 * \delta_2$ of Δ on $A \coprod B \coprod G_{\Gamma_0}$ is defined:

- on A by δ_1,
- on B by $\delta_2^z\colon B \to B^z; y_i \to \delta_2(y_i)^z$, where B^z is the group generated y_j^z,
- on G_{Γ_0} by inversion.

We now deduce results on the structure of G_L.

Corollary 2.16. — *We have $G_L \simeq G_{L_1} \coprod G_{L_2} \coprod G_{\Gamma_0}$.*

Proof. — Theorem 2.6 gives us $G_K \simeq G_L \rtimes_{\psi_K} \Delta$. Remark 2.15 allows us to conclude that $G_K \simeq (G_{L_1} \coprod G_{L_2} \coprod G_{\Gamma_0}) \rtimes_{\psi_{K_1} * \psi_{K_2}} \Delta$. We observe that G_{L_1}, G_{L_2} and G_{Γ_0} do not contain involutions. Thus $G_{L_1} \coprod G_{L_2} \coprod G_{\Gamma_0}$ does not contain involutions. Therefore, we conclude from Theorem 2.6 that $G_L \simeq G_{L_1} \coprod G_{L_2} \coprod G_{\Gamma_0}$. □

2.4. Conclusion. — We are now able to prove Theorem 2.1.

Proof Theorem 2.1. — We show (ii) implies (i). If G_K is Δ-RAAG then, from Theorem 2.6, the actions ψ_K and $\delta_{\mathbf{z}}$ coincide. Indeed G_Γ does not have an involution, so $G_L = G_\Gamma$ also does not. Proposition 2.11 and Theorem 2.13 show that the class of Δ-RAAGs satisfies the conditions of Theorem 2.10. Thus, by Theorem 2.10, we conclude that $\mathscr{P}$ is a subclass of Δ-RAAGs.

Conversely, let us show (i) implies (ii). Assume that there exists a graph Γ and an element $\mathbf{z}$ in $G_\Gamma^{d_\Gamma}$ such that $G_K \simeq G_{\mathbf{z}}$. Let $K^{(3)}$ be the compositum of all Galois extensions of degree dividing 4 of K. From [36, Proposition 2.1], we infer that $G_{K,3} = G_{K^{(3)}}$. From [35, Theorems 2.7 and 2.11] and Corollary 1.8, we deduce that K is RPF.

The last part of our result is given by Corollary 1.5. □

Example 2.17. — Let us give a few examples.

• For every integer d_Γ, the groups $G_K \coloneqq G_{\Gamma^c} \rtimes_{\delta_0} \Delta$ (superpythagorean) and $G_K \coloneqq G_{\Gamma^f} \rtimes_{\delta_0} \Delta$ (SAP) are in $\mathscr{P}$.

• From Theorem 2.10, the group $G_{\mathbf{z}_0}$ is in $\mathscr{P}$ for every graph Γ with at most 3 vertices.

• The following graph Γ, described by four vertices and two disjoint edges, gives the group $G \coloneqq G_\Gamma \rtimes_{\delta_{\mathbf{z}_0}} \Delta \simeq (\mathbb{Z}_2^2 \coprod \mathbb{Z}_2^2) \rtimes_{\delta_{\mathbf{z}_0}} \Delta$:

$$\begin{tikzcd} x_1 \arrow[d, dash] & x_3 \arrow[d, dash] \\ x_2 & x_4 \end{tikzcd}$$

From the results in the subsection 2.3, the previous group is not in $\mathscr{P}$. However, [45, Theorem 2] shows that the group G_Γ is realizable as a group G_L, for some field L containing a square root of -1.

Remark 2.18. — If K is a RPF field, then the group G_K is uniquely determined by an underlying graph Γ and an element $\mathbf{z}$ in $G_\Gamma^{d_\Gamma}$. In the next section, we prove a result stronger than Theorem 2.1: Theorem 3.4. This Theorem shows that the element $\mathbf{z}$ in $G_\Gamma^{d_\Gamma}$ is uniquely determined by the graph Γ for RPF fields.

3. Δ-RAAG theory and Pythagorean fields

This section investigates $Gr(\mathscr{P})$. This is the class of graphs which are underlying graphs of groups in $\mathscr{P}$. Let Γ be in $Gr(\mathscr{P})$. From [45, Theorem 1.2], the graph Γ does not contain as subgraphs the square graph C_4 and the following line L_3:

$$i_1 \text{ ——— } i_2 \text{ ——— } i_3 \text{ ——— } i_4$$

It is also shown that $H^\bullet(G_\Gamma)$ is (universally) Koszul, and G_Γ is absolutely torsion-free. However, this condition is not sufficient to characterize $Gr(\mathscr{P})$. We refer to Example 2.17.

3.1. Structure of $Gr(\mathscr{P})$. —

We start this section by giving examples of graphs in $Gr(\mathscr{P})$.

Corollary 3.1. — *Let (n, m) be a couple of positive integers. We define Γ^{cn} the complete graph on n vertices, and Γ^{fm} the free graph on m vertices. Then $\Gamma^{cn}\coprod\Gamma^{fm}$ and $\Gamma^{cn}\nabla\Gamma^{fm}$ are both in $Gr(\mathscr{P})$.*

Precisely, the following groups are realizable as groups in $\mathscr{P}$:

(i) $G_{(\Gamma^{cn}\coprod\Gamma^{fm}),\mathbf{z}_0}$,
(ii) $G_{(\Gamma^{cn}\nabla\Gamma^{fm}),\mathbf{z}_0}$, *where ∇ denotes the join of two graphs.*

Proof. — The groups $G_{\Gamma^{cn},\mathbf{z}_0}$ and $G_{\Gamma^{fm},\mathbf{z}_0} \simeq \Delta\coprod\dots\coprod\Delta$ are realizable as pro-2 quotients of absolute Galois groups over superpythagorean and SAP fields. Observe from Theorem 2.13 and Proposition 2.11 that:

$$G_{(\Gamma^{cn}\coprod\Gamma^{fm}),\mathbf{z}_0} \simeq \Delta\coprod\dots\coprod\Delta\coprod G_{\Gamma^{cn},\mathbf{z}_0}, \quad \text{and} \quad G_{(\Gamma^{cn}\nabla\Gamma^{fm}),\mathbf{z}_0} = (G_{\Gamma^{cn}}\times G_{\Gamma^{fm}})\rtimes_{i_n\times\delta_{\mathbf{z}_0'}}\Delta,$$

with $\mathbf{z}_0' \coloneqq (1,\dots,1) \in G^m_{\Gamma^{fm}}$. We conclude using Theorem 2.10. $\square$

We now study the structure of $Gr(\mathscr{P})$. Consider Γ in $Gr(\mathscr{P})$. We denote by m_0 and m the number of isolated vertices and nontrivial connected components of Γ. We introduce Γ_i, the nontrivial connected components of Γ, with $1 \leqslant i \leqslant m$. Then from Corollary 2.16, we can write:

$$\Gamma \coloneqq \coprod_{i=1}^{m}\Gamma_i\coprod_{j=1}^{m_0}\Gamma_0,$$

with Γ_0 a graph with one vertex. This gives us: $G_\Gamma \simeq \coprod_{i=1}^m G_{\Gamma_i}\coprod_{j=1}^{m_0} G_{\Gamma_0}$.

Lemma 3.2. — *Assume that Γ is in $Gr(\mathscr{P})$. Then for every $1 \leqslant i \leqslant l$, the graph Γ_i is in $Gr(\mathscr{P})$. Additionally $m_0 \geqslant m-1$.*

Concretely, there exists RPF fields K_i with underlying graphs Γ_i, for $1 \leqslant i \leqslant m$, such that:

$$G_K \simeq \coprod_{i=1}^{m} G_{K_i} \coprod_{j=1}^{m_0-m+1} G_{\Gamma_0},$$

where K is in $\mathscr{P}$ and has underlying graph Γ.

Proof. — Assume that Γ is disconnected. Then from Theorem 2.10 and Corollary 2.16, there exist two RPF fields K_a and K_b with underlying graphs Γ_a and Γ_b such that:

$$G_K \simeq \mathrm{G}_{K_a} \coprod G_{K_b}, \quad G_\Gamma \simeq G_{\Gamma_a} \coprod G_{\Gamma_b} \coprod G_{\Gamma_0}.$$

From Lemma 1.4, we can write $\Gamma \simeq \Gamma_a \coprod \Gamma_b \coprod \Gamma_0$. Applying the previous argument to Γ_a and Γ_b, we can find RPF fields $K_{u_1}, \ldots, K_{u_{m'}}$ with underlying connected graphs $\Gamma_{u_1}, \ldots, \Gamma_{u_{m'}}$ such that:

$$G_K \simeq \coprod_{i=1}^{m'} G_{K_{u_i}} \quad G_L = \coprod_{i=1}^{m'} G_{\Gamma_{u_i}} \coprod_{j=1}^{m'-1} G_{\Gamma_0}.$$

Applying Lemma 1.4, this gives us the decomposition:

$$\Gamma \simeq \coprod_{i=1}^{m'} \Gamma_{u_i} \coprod_{j=1}^{m'-1} \Gamma_0.$$

We observe that m is the number of graphs Γ_{u_i} with more than two vertices. So $m' \geqslant m$. Thus $m_0 \coloneqq 2(m'-m)+m-1 \geqslant m-1$. This allows us to conclude. □

Lemma 3.3. — *Assume that Γ is connected and in $Gr(\mathscr{P})$. Then there exists an integer u_Γ and a RPF field K' with a unique disconnected underlying graph Γ' such that:*

$$G_K \simeq G_{\Gamma^{cu_\Gamma}} \rtimes_{i_{u_\Gamma}} G_{K'},$$

where Γ^{cu_Γ} is the complete graph on u_Γ vertices. Consequently, the underlying graph of G_K is $\Gamma \coloneqq \Gamma^{cu_\Gamma} \nabla \Gamma'$.

Proof. — From Proposition 2.11 and Corollary 2.12, there exists a RPF field K_a with underlying field Γ_a such that:

$$G_K \simeq G_{\Gamma_0} \rtimes_i G_{K_a}, \quad G_\Gamma \simeq G_{\Gamma_0} \times G_{\Gamma_a}.$$

Applying the same argument to G_{K_a}. We infer an integer u_Γ and a RPF field K' with disconnected underlying graph Γ' such that:

$$G_K \simeq G_{\Gamma^{cu_\Gamma}} \rtimes_{i_{u_\Gamma}} G_{K'}, \quad G_\Gamma \simeq G_{\Gamma^{cu_\Gamma}} \times G_{\Gamma'}.$$

From Lemma 1.4, we infer $\Gamma \simeq \Gamma^{cu_\Gamma} \nabla \Gamma'$. □

From Lemmata 3.3 and 3.2, we can show the converse of Theorem 2.1. This allows us to conclude the proof of Theorem A.

Theorem 3.4. — *Let us consider two RPF fields K_1 and K_2, with underlying graphs Γ_1 and Γ_2. If $\Gamma_1 \simeq \Gamma_2$, then $G_{K_1} \simeq G_{K_2}$. Consequently, Δ-RAAGs in $\mathscr{P}$ are uniquely determined by their underlying graphs.*

Proof. — We define Γ the underlying graph of G_{K_1} and G_{K_2}. We show by induction on d_Γ that $G_{K_1} \simeq G_{K_2}$.

If $d_\Gamma = 0$, then $G_{K_1} \simeq G_{K_2} \simeq \Delta$.

If $d_\Gamma = 1$, then $G_{K_1} \simeq G_{K_2} \simeq G_{\Gamma_0} \rtimes_{\mathbf{z}_0} \Delta$. Both groups are SAP and superpythagorean.

If $d_\Gamma > 1$, we distinguish two cases:

• Assume that the graph Γ is connected. Then from Lemma 3.3, there exists an integer $u_\Gamma > 0$ and two RPF fields K_1' and K_2' with same disconnected underlying graph Γ' such that:

$$G_{K_1} \simeq G_{\Gamma^{cu_\Gamma}} \rtimes_{i_{u_\Gamma}} G_{K_1'}, \quad G_{K_2} \simeq G_{\Gamma^{cu_\Gamma}} \rtimes_{i_{u_\Gamma}} G_{K_2'}.$$

Since $d_{\Gamma'} < d_\Gamma$, we deduce, by induction hypothesis, that $G_{K_1'} \simeq G_{K_2'}$. Thus $G_{K_1} \simeq G_{K_2}$.

• If we are not in the previous case, then the graph Γ is disconnected. Following Lemma 3.2, there exist RPF fields K_{1i} and K_{2i} with underlying graphs Γ_i, for $1 \leqslant i \leqslant m$ such that:

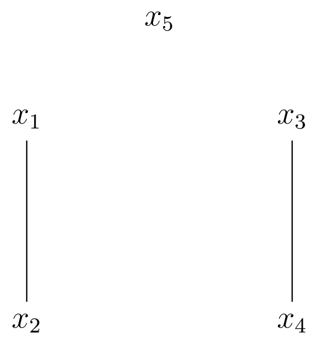
$$G_{K_1} \simeq \coprod_{i=1}^{m} G_{K_{1i}} \coprod_{j=1}^{m_0-m-1} G_{\Gamma_0}, \quad \text{and} \quad G_{K_2} \simeq \coprod_{i=1}^{m} G_{K_{2i}} \coprod_{j=1}^{m_0-m-1} G_{\Gamma_0}.$$

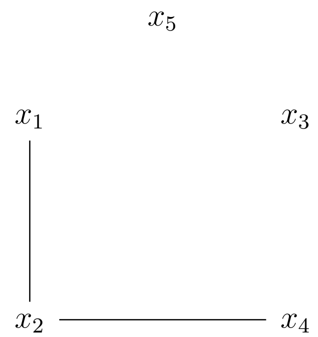
Since for every $1 \leqslant i \leqslant m$ we have $d_{\Gamma_i} < d_\Gamma$, we deduce by induction hypothesis that $G_{K_{1i}} \simeq G_{K_{2i}}$. Thus $G_{K_1} \simeq G_{K_2}$. □

Example 3.5. — Let us consider the groups

$$G_1 \coloneqq (\mathbb{Z}_2^2 \rtimes_{\delta_{\mathbf{z}_0}} \Delta) \coprod (\mathbb{Z}_2^2 \rtimes_{\delta_{\mathbf{z}_0}} \Delta), \quad \text{and} \quad G_2 \coloneqq (\mathbb{Z}_2 \rtimes_i (F(2) \rtimes_{\delta_{\mathbf{z}_0}} \Delta)) \coprod \Delta \coprod \Delta.$$

These groups are Δ-RAAGs, and in $\mathscr{P}$. We can represent them with the following graphs:

$\Gamma_1 \coloneqq$ 

$\Gamma_2 \coloneqq$ 

Precisely, if we define

$$\mathbf{z}_1 \coloneqq (1,1,x_5,x_5,1) \in G_{\Gamma_1}^5, \quad \text{and} \quad \mathbf{z}_0 \coloneqq (1,1,1,1,1) \in G_{\Gamma_2}^5, \quad \text{and} \quad \mathbf{z}_0' \coloneqq (1,1,1,1,1) \in G_{\Gamma_1}^5.$$

we infer $G_1 \coloneqq G_{\Gamma_1} \rtimes_{\delta_{\mathbf{z}_1}} \Delta$ and $G_2 \coloneqq G_{\Gamma_2} \rtimes_{\delta_{\mathbf{z}_0}} \Delta$. We observe that G_1 and G_2 have the same Poincaré and gocha series, but are not isomorphic, since Γ_1 and Γ_2 are not. We refer to Theorem 2.1.

Let us now consider the group $G_3 \coloneqq G_{\Gamma_1} \rtimes_{\delta_{\mathbf{z}_0'}} \Delta$. We observe that G_1 and G_3 are not isomorphic. Thus from Theorem 3.4, the group G_3 is not in $\mathscr{P}$. But of course G_3 has the same gocha series as G_1 and G_2.

3.2. Koszulity, cohomology and gocha series for groups in $\mathscr{P}$. — Let Γ be in $Gr(\mathscr{P})$. In this subsection we study $\Gamma(t)$. Let us also refer to [49] for a study of clique polynomials in other contexts.

We start this subsection by answering to a question asked by Weigel in [48] for the class $\mathscr{P}$. *If G is in $\mathscr{P}$, is $\mathscr{E}(G)$ Koszul?*

Proposition 3.6. — *If G is in $\mathscr{P}$, then $\mathscr{E}(G)$ is Koszul. Consequently, there exists a graph Γ and a vector $\mathbf{z} \in G_\Gamma^{d_\Gamma}$ such that $G \simeq G_{\mathbf{z}}$. Furthermore, the presentation* ($\mathbf{z}$-Pres) *is minimal and*

$$d(G) := \dim_{\mathbb{F}_2} H^1(G) = d_\Gamma + 1 \quad \textit{and} \quad r(G) := \dim_{\mathbb{F}_2} H^2(G) = d_\Gamma + r_\Gamma + 1.$$

Additionally, the Poincaré series of G is given by:

$$H^\bullet(G,t) := \sum_n \dim_{\mathbb{F}_2} H^n(G) t^n = \frac{\Gamma(t)}{1-t}.$$

Proof. — Take G in $\mathscr{P}$. From [32, Theorem $F,(3)$] or [33, Theorem $D,(g)$], we observe that $H^\bullet(G)$ is Koszul. From Theorem 2.1, there exists $\mathbf{z}$ and Γ such that $G_{\mathbf{z}} \simeq G$. Then from Remark 1.11 and [33, Theorem F], we conclude that $H^\bullet(G)$ is the quadratic dual of $\mathscr{E}(G)$. Consequently $\mathscr{E}(G)$ is also Koszul. We infer from Corollary 1.12 that:

$$\begin{aligned} H^\bullet(G,t) &:= \sum_n \dim_{\mathbb{F}_2} H^n(G) t^n = \frac{\Gamma(t)}{1-t} \\ &= (1 + d_\Gamma t + r_\Gamma t^2 + \dots)(1 + t + t^2 + \dots) = 1 + (d_\Gamma + 1)t + (r_\Gamma + d_\Gamma + 1)t^2 + \dots \end{aligned}$$

Thus $d(G) = d_\Gamma + 1$, and $r(G) = d_\Gamma + r_\Gamma + 1$. □

We conclude this subsection by using results from [31] to characterise which polynomials $\Gamma(t)$ are realizable, for Γ in $Gr(\mathscr{P})$.

Theorem 3.7. — *Let Γ be an undirected graph. The graph Γ is in $Gr(\mathscr{P})$, if and only if there exists integers s and $a_0, \dots, a_{s-1}$ satisfying all of the following conditions:*

(i) $0 \leqslant a_0, \dots, a_{s-2}$,
(ii) $1 \leqslant a_{s-1}, s$,
(iii) $a_0 + \dots + a_{s-1} + s = d_\Gamma$,
(iv) $\Gamma(t) = (1+t)^{s-1} + t\left((1+t)^{s-1} a_{s-1} + (1+t)^{s-2} a_{s-2} + \dots + a_0\right)$.

Proof. — The Poincaré series of G_Γ is given by:

$$H^\bullet(G_\Gamma, t) := \sum_{n \in \mathbb{N}} \dim_{\mathbb{F}_2} H^n(G_\Gamma) t^n = \Gamma(t).$$

Let Γ be in $Gr(\mathscr{P})$. From [31, Theorem 11], we conclude that $\Gamma(t) = (1+t)^{s-1} + t\left((1+t)^{s-1} a_{s-1} + \dots + a_0\right)$ for some integers s, $a_0, \dots, a_{s-1}$ satisfying $(i) - (iii)$.

The converse is a consequence of [31, Theorem 11]. □

4. An interesting example

We conclude this paper with an example of a pro-2 group which is not a maximal pro-2 quotient of an absolute Galois group, but satisfies the Koszul, the strong Massey Vanishing and the Kernel Unipotent properties. Let us first recall these properties for a general pro-p group G.

• We say that a pro-p group G satisfies the Koszul property if the cohomology ring $H^\bullet(G; \mathbb{F}_p)$ is Koszul. Positselski [41] formulated the following conjecture. All maximal pro-p quotients of absolute Galois groups over fields, containing the primitive p-th roots of unity, satisfy the Koszul property.

• We recall the Kernel Unipotent property. We fix a positive integer n, and define:

$$G_{\langle n\rangle} := \bigcap_{\rho\colon G\to \mathbb{U}_n(\mathbb{F}_p)} \ker\rho.$$

We easily observe that $G_n \subset G_{\langle n\rangle}$, and we say that G satisfies the n-Kernel Unipotent property if $G_{\langle n\rangle} = G_n$. If G satisfies the n-Kernel Unipotent property for every n, we say that G satisfies the Kernel Unipotent property. Efrat and the last two authors [37, Appendix] gave, for every integer $n \geqslant 3$, examples of groups which are not maximal pro-p quotients of absolute Galois groups and do not satisfy the n-Kernel Unipotent property. The last two authors also formulated the following conjecture [37, Conjecture 1.3]. All maximal pro-p quotients of absolute Galois groups over fields, containing the primitive p-th root of unity, satisfy the Kernel Unipotent property.

• We recall the strong Massey Vanishing property. Let n be a positive integer. Let α be a family of characters $\alpha := \{\alpha_i\colon G \to \mathbb{F}_p\}_{i=1}^n$ in $H^1(G;\mathbb{F}_p)^n$ checking $\alpha_i \cup \alpha_{i+1} = 0$. We say that a group G satisfies the strong Massey Vanishing property if for every n and α, there exists a morphism $\rho\colon G \to \mathbb{U}_{n+1}(\mathbb{F}_p)$ such that for every g in G, we have

$$\rho_{i,i+1}(g) := \rho(g)_{i,i+1} = \alpha_i(g).$$

The strong Massey Vanishing property does not hold for all maximal pro-p quotients of absolute Galois groups of fields containing a p-th root of unity. We refer to Harpaz-Wittenberg's counterexample [8, Example A.15]. However, Ramakrishna and the last three authors [21, Theorem 1] showed this property for maximal pro-p (tame) quotients of absolute Galois groups of number fields, which do not contain a primitive p-th root of unity. Quadrelli [44]- [43] showed the strong Massey Vanishing property for Elementary pro-p groups and the class $\mathscr{P}$. Observe also that the strong Massey Vanishing property implies the Massey Vanishing property stated by the last two authors. We refer to [23] for further details.

This section aims to prove Theorem B. Let us define the pro-2 group G by the presentation:

$$(\mathbf{z}_0\text{-Pres})\quad G := \langle x_0, x_1, x_2, x_3, x_4 |\quad [x_1, x_2] = [x_2, x_3] = [x_3, x_4] = [x_4, x_1] = 1,$$
$$x_0^2 = 1, x_0x_jx_0x_j = 1, \forall j \in [\![1; 4]\!]\rangle$$

Theorem 4.1. — *The group G is not a maximal pro-2 quotient of an absolute Galois group but satisfies the Koszul, the Kernel Unipotent and the strong Massey Vanishing properties. Furthermore, the presentation* ($\mathbf{z}_0$-Pres) *is minimal and $\mathscr{E}(G)$ is Koszul.*

We introduce several pro-2 groups which will play a fundamental role in the study of G and the proof of Theorem 4.1. Let G_{13} and G_{24} be the subgroups of G generated by $\{x_0, x_1, x_3\}$ and $\{x_0, x_2, x_4\}$. We also introduce F_{13} and F_{24} the subgroups of G generated by $\{x_1, x_3\}$ and $\{x_2, x_4\}$. We define $G_0 := \langle y_1\rangle \coprod \langle y_2\rangle \coprod \langle y_3\rangle \simeq \Delta \coprod \Delta \coprod \Delta$, where $y_i^2 = 1$, for $1 \leqslant i \leqslant 3$. The group G_0 is a quotient of G and has an easy structure. It is SAP, so in $\mathscr{P}$.

Furthermore, we easily observe that we have natural surjections $\pi_{13}\colon G_0 \to G_{13}$ and $\pi_{24}\colon G_0 \to G_{24}$ defined by:

$$\pi_{13}(y_1y_2) := x_1,\quad \pi_{13}(y_1y_3) := x_3, \pi_{24}(y_1y_2) := x_2,\quad \pi_{24}(y_1y_3) := x_4\quad \pi_{13}(y_1) = \pi_{24}(y_1) := x_0.$$

4.1. First properties of the group G. — We observe that G is a Δ-RAAG given by $G \coloneqq (F(2) \times F(2)) \rtimes_{\delta_{\mathbf{z}_0}} \Delta$. The underlying graph of the Δ-RAAG G is the square graph $\Gamma = C_4$:

$$\begin{tikzcd} x_1 \arrow[r, dash] \arrow[d, dash] & x_2 \arrow[d, dash] \\ x_4 \arrow[r, dash] & x_3 \end{tikzcd}$$

And the action is defined by the vector $\mathbf{z}_0 \coloneqq (1,1,1,1)$ in G_Γ^4. Thanks to the Δ-RAAG theory, we infer the following result:

Lemma 4.2. — *The group G is not a maximal pro-2 quotient of an absolute Galois group.*

Proof. — The group G_Γ is not realizable as a maximal pro-2 quotient of an absolute Galois group of a field containing a second root of unity. We refer to [45, Theorem 1.2] or [42, Theorem 5.6]. Thus G is not in $\mathscr{P}$. Alternatively, we observe from Lemmata 3.3 and 3.2 that Γ is not in $Gr(\mathscr{P})$. Thus G is not in $\mathscr{P}$, and so not a maximal pro-2 quotient of an absolute Galois group. □

Remark 4.3. — We observe that Γ is the only graph with clique polynomial $\Gamma(t) \coloneqq 1 + 4t + 4t^2$. Thus from Theorem 3.7, we conclude that there exists no Δ-RAAG G' in $\mathscr{P}$ with underlying graph Γ' such that $\Gamma(t) = \Gamma'(t)$.

Despite not being in $\mathscr{P}$, the group G contains subgroups in $\mathscr{P}$.

Lemma 4.4. — *The groups G_{13} and G_{24} are isomorphic to G_0. The groups F_{13} and F_{24} are pro-2 free on $\{x_1, x_3\}$ and $\{x_2, x_4\}$. Additionally, we have the decomposition*

$$G \simeq F_{24} \rtimes_{\delta_F} G_{13} \simeq F_{13} \rtimes_{\delta_{F'}} G_{24},$$

where the action δ_F and $\delta_{F'}$ are defined by:

$$\begin{aligned} \delta_F(x_0)(x_i) &= x_i^{-1}, \quad \delta_F(x_j)(x_i) = x_i, \quad \text{for } i \in \{2,4\}, \text{ and } j \in \{1,3\}, \\ \delta_{F'}(x_0)(x_j) &= x_j^{-1}, \quad \delta_{F'}(x_i)(x_j) = x_j, \quad \text{for } i \in \{2,4\}, \text{ and } j \in \{1,3\}. \end{aligned}$$

Proof. — Let $F(2)$ be the pro-2 group on two generators a and b. We define an action ψ_0 of G_0 on $F(2)$ by $\psi_0(y_i)(a) \coloneqq a^{-1}$ and $\psi_0(y_i)(b) \coloneqq b^{-1}$ for $1 \leqslant i \leqslant 3$. Similarly to Remark 2.9, we observe that $G \simeq F(2) \rtimes_{\psi_0} G_0$. Therefore, we obtain two injections φ_{13} and φ_{24} from G_0 to G defined by:

$$\varphi_{ij}(y_1 y_2) = x_i, \quad \varphi_{ij}(y_1 y_3) = x_j, \quad \text{and} \quad \varphi_{ij}(y_1) = x_0.$$

We also introduce the natural injections $\iota_{13} \colon G_{13} \to G$ and $\iota_{24} \colon G_{24} \to G$. We infer the following commutative diagram:

$$\begin{tikzcd} & G_{24} \arrow[dr, hook, "\iota_{24}"] & \\ G_0 \arrow[ur, two heads, "\pi_{24}"] \arrow[rr, hook, "\varphi_{24}", "\varphi_{13}"'] \arrow[dr, two heads, "\pi_{13}"'] & & G \\ & G_{13} \arrow[ur, hook, "\iota_{13}"'] & \end{tikzcd}$$

Since φ_{13} and φ_{24} are both injective, the maps π_{13} and π_{24} are isomorphisms. Similarly, we show that $F_{13} \simeq F_{24} \simeq F(2)$. This allows us to conclude. $\square$

As a consequence, we infer that the subgroup of G generated by $\{x_1, x_2, x_3, x_4\}$ is isomorphic to $F_{13} \times F_{24}$, which is exactly G_Γ.

4.2. Cohomological results on G. — We now study the cohomology ring $H^\bullet(G)$.

Lemma 4.5. — *The algebra $\mathscr{E}(G)$ is Koszul. As a consequence, the presentation* ($\mathbf{z}_0$-Pres) *is minimal and the group G satisfies the Koszul property.*

Proof. — Let us consider $\mathscr{E}$ the noncommutative polynomials on $\{X_0, X_1, X_2, X_3, X_4\}$ over $\mathbb{F}_2$ where each X_i has weight 1. A graded $\mathbb{F}_2$-basis of $\mathscr{E}$ is given by monomials, and we endow them with an order induced by $x_0 > x_1 > x_3 > x_2 > x_4$. To show that the algebra $\mathscr{E}(G)$ is Koszul, it is sufficient to show that it is PBW. This property comes from Poincaré-Birkhoff-Witt, and we recall it briefly. We refer to [20, §4.1] or [33]. From Theorem 1.10, we have $\mathscr{E}(G) \simeq \mathscr{E}_{\mathbf{z}_0}$. Thus $\mathscr{E}(G)$ is presented by 5 generators $\{X_0, X_1, X_2, X_3, X_4\}$ and 9 relations $\mathscr{R} \coloneqq \{X_0^2, [X_u, X_v]; [X_0, X_i] + X_i^2\}$, with $u \in \{1, 3\}$, $v \in \{2, 4\}$ and $1 \leqslant i \leqslant 4$. The leading monomials of these relations are given by:

$$M(\mathscr{R}) \coloneqq \{X_0^2, X_1X_2, X_3X_2, X_3X_4, X_1X_4, X_0X_1, X_0X_2, X_0X_3, X_0X_4\}.$$

A monomial is said to be reduced if it does not contain any leading monomial as a submonomial. We define critical monomials by monomials of the form $X_uX_vX_w$ where X_uX_v and X_vX_w are in $M(\mathscr{R})$. In our example, the critical monomials are exactly

$$C(\mathscr{R}) \coloneqq \{X_0^2X_i, X_0X_uX_v, \quad 1 \leqslant i \leqslant 4, \text{ and } u \in \{1, 3\}, v \in \{2, 4\}\}.$$

Each relation in $\mathscr{R}$ can be interpreted as a reduction rule which substitutes its leading monomial in $M(\mathscr{R})$ to a sum of lower monomials. There are two ways to write a critical monomial as a sum of reduced monomials, with respect to reduction rules. This is technical; however, we can at least say that the first way is to reduce it from the left to the right, and the second way is to reduce it from the right to the left. For instance, we refer to [33, §2.4] or [20, §4.3.5] for further details. These reductions allow us to construct, for each critical monomial, a diagram. This diagram starts with a vertex given by a critical monomial and ends with at most two vertices. Each ending vertex is given by a sum of reduced monomials. We say that a critical monomial is confluent if its associated diagram finishes with one vertex, i.e. the two sums of reduced monomials coincide. The algebra $\mathscr{E}(G)$ is PBW if every critical monomial is confluent.

In our example, we observe that every critical monomial is confluent, so $\mathscr{E}(G)$ is PBW. For instance, we apply the previous reduction rules to the critical monomial $X_0X_1X_2$. We infer the following diagram:

$$
\begin{tikzcd}
 & X_0X_1X_2 \arrow[dl] \arrow[dr] & \\
(X_1X_0+X_1^2)X_2 \arrow[dd] & & X_0X_2X_1 \arrow[d] \\
 & & (X_2X_0+X_2^2)X_1 \arrow[d] \\
X_1(X_2X_0+X_2^2)+X_1^2X_2 \arrow[dr] & & X_2(X_1X_0+X_1^2)+X_2^2X_1 \arrow[dl] \\
 & X_2X_1X_0+X_2^2X_1+X_2X_1^2 &
\end{tikzcd}
$$

From [11, Proposition 1] or [19], we deduce that the group G satisfies the Koszul property. As a consequence of Corollary 1.12, the gocha and Poincaré series of G are given by:

$$gocha(G,t) \coloneqq \frac{1+t}{1-4t+4t^2}, \quad \text{and} \quad H^\bullet(G,t) \coloneqq \sum_n \dim_{\mathbb{F}_2} H^n(G)t^n = \frac{1+4t+4t^2}{1-t}.$$

Therefore the presentation ($\mathbf{z}_0$-Pres) is minimal. $\square$

Let us denote by χ_0 (resp. ψ_i) the character associated to x_0 (resp. x_i).

Lemma 4.6. — *The algebra $H^\bullet(G)$ is the quadratic dual of $\mathscr{E}(G)$. Concretely, the algebra $H^\bullet(G)$ is generated by the generators χ_0 and ψ_i for $1 \leqslant i \leqslant 4$, and relations:*

$$\chi_0 \cup \psi_i + \psi_i \cup \chi_0, \quad \psi_i \cup \psi_j + \psi_j \cup \psi_i, \quad \chi_0 \cup \psi_i + \psi_i^2, \quad \psi_1 \cup \psi_3, \quad \psi_2 \cup \psi_4,$$

with $1 \leqslant i < j \leqslant 4$. Thus a basis of $H^2(G)$ is given by:

$$\chi_0^2, \quad \chi_0 \cup \psi_i = \psi_i^2, \quad \psi_1 \cup \psi_2, \quad \psi_1 \cup \psi_4, \quad \psi_3 \cup \psi_2, \quad \psi_3 \cup \psi_4.$$

Proof. — In Lemma 4.5, we showed that the algebra $\mathscr{E}(G)$ is Koszul. Thus $H^\bullet(G)$ is the quadratic dual of $\mathscr{E}(G)$. We refer to [11, Proposition 1]. We compute the quadratic dual of $\mathscr{E}(G)$. We refer to [20, Chapter 3, §3.2.2] for further details.

By a dimension argument, we observe that the set

$$\{\chi_0 \cup \psi_i + \psi_i \cup \chi_0, \quad \psi_i \cup \psi_j + \psi_j \cup \psi_i, \quad \chi_0 \cup \psi_i + \psi_i^2, \quad \psi_1 \cup \psi_3, \quad \psi_2 \cup \psi_4\}$$

defines all relations for the quadratic dual of $\mathscr{E}(G)$. We conclude that the quadratic dual of $\mathscr{E}(G)$ has a presentation with generators χ_0, ψ_i and relations defined above. $\square$

Lemma 4.6 allows us to infer:

Corollary 4.7. — *We have the following isomorphisms*

$$\begin{aligned} H^1(G) \simeq& H^1(G_{13}) \bigoplus H^1(F_{24}) \simeq H^1(G_{24}) \bigoplus H^1(F_{13}), \\ H^2(G) \simeq& H^2(G_{13}) \bigoplus \left(H^1(G_{13}) \cup \psi_2\right) \bigoplus \left(H^1(G_{13}) \cup \psi_4\right), \end{aligned}$$

where we identify

- *$H^1(G_{ij})$ and $H^1(F_{ij})$ with the subgroups of $H^1(G)$ generated by $\{\chi_0, \psi_i, \psi_j\}$ and $\{\psi_i, \psi_j\}$,*
- *$H^2(G_{13})$ with the subgroup of $H^2(G)$ generated by χ_0^2, $\chi_0 \cup \psi_1 = \psi_1^2$ and $\chi_0 \cup \psi_3 = \psi_3^2$.*

4.3. The Kernel Unipotent property. — In this subsection, we show that G satisfies the Kernel Unipotent property. It is strongly inspired by [37, §2].

We begin with the following lemma:

Lemma 4.8. — *The group G_0 satisfies the Kernel Unipotent property.*

Proof. — For every positive integer n and every $g \in G_{0,n-1} \backslash G_{0,n}$, we construct a map $\rho_g \colon G_0 \to \mathbb{U}_n$ such that $\rho_g(g) \neq \mathbb{I}_n$.

Let us recall that $G_0 \simeq \Delta \coprod \Delta \coprod \Delta := \langle y_1 \rangle \coprod \langle y_2 \rangle \coprod \langle y_3 \rangle$, where y_i denotes an involution. Let us define E by the set of noncommutative series on $\mathbb{F}_2$ over Y_1, Y_2 and Y_3, and I the closed two-sided ideal of E generated by $\{Y_1^2, Y_2^2, Y_3^2\}$. Here, every Y_1, Y_2 and Y_3 have weight 1. From the Magnus isomorphism, we infer the isomorphism $\psi_{G_0} \colon E(G_0) \simeq E/I$, which maps y_i to $1 + Y_i$. In particular, if $g \in G_{0,n-1} \backslash G_{0,n}$, then we can write:

$$\psi_{G_0}(g) := 1 + \sum_W \epsilon_W(g) W + W_{\geqslant n},$$

for $\epsilon_W(g)$ in $\mathbb{F}_2$, $W := Y_{i_1} Y_{i_2} \dots Y_{i_{n-1}}$ words satisfying $i_j \neq i_{j+1}$ with $1 \leqslant j \leqslant n-2$, and $W_{\geqslant n+1}$ a series of degree larger than $n+1$. The previous form is unique.

We introduce $\mathbb{M}_n$, the set of $n \times n$ matrices with coefficients in $\mathbb{F}_2$. We denote by $\delta_{i,j}$ the elementary $n \times n$-matrix, which is equal to zero everywhere except in (i,j). Let us define a morphism ρ_W from $E(G_0)$ to $\mathbb{M}_n$ satisfying $\rho_W(W) = \delta_{1,n}$.

We fix a word $W := Y_{i_1} Y_{i_2} \dots Y_{i_{n-1}}$ in $E(G_0)$ satisfying $i_j \neq i_{j+1}$. And for every $1 \leqslant i \leqslant 3$, we define a map $\psi_i \colon [\![1; n-1]\!] \to \mathbb{F}_2$ by:

$$\psi_i(j) := 1, \text{ if } Y_{i_j} = Y_i, \quad \text{else } \psi_i(j) := 0.$$

We introduce the matrix

$$M_i := \sum_{j=1}^{n-1} \psi_i(j) \delta_{j,j+1}.$$

Since for every j, we have $Y_{i_j} \neq Y_{i_{j+1}}$, we observe for every $1 \leqslant i \leqslant 3$ the equality $M_i^2 = 0$.

Thus we define a morphism ρ_W by $\rho_W(1) = \mathbb{I}_n$ and $\rho_W(Y_i) := M_i$. If $W' := Y_{u_1} \dots Y_{u_{n-1}}$ is a word also satisfying $u_j \neq u_{j+1}$, then we have:

$$\rho_W(W') = \psi_{u_1}(1) \dots \psi_{u_{n-1}}(n-1) \delta_{1,n}.$$

In particular $\rho_W(W') = \delta_{1,n}$ only if $W' = W$, else $\rho_W(W') = 0$.

Let us now fix a nontrivial element g in $G_{0,n-1}/G_{0,n}$. Then there exists a word W of degree $n-1$ such that $\epsilon_W(g) \neq 0$. We define ρ_g the map induced by ρ_W, i.e. $\rho_g(y_i) := \mathbb{I}_n + M_i$. We observe that $\rho_g(y_i)^2 = \mathbb{I}_n^2 + M_i^2 = \mathbb{I}_n$, and $\rho_g(Y_i) = \rho_W(Y_i)$. Thus we have:

$$\rho_g(g) = \mathbb{I}_n + \sum_{W'} \epsilon_{W'}(g) \rho_g(W') = \mathbb{I}_n + \epsilon_W(g) \rho_W(W) = \mathbb{I}_n + \delta_{1,n} \neq \mathbb{I}_n,$$

where the sum is indexed by all monomials W' in E of degree n. □

Remark 4.9. — Without loss of generality, we can expand the proof of Lemma 4.8 and show that if G is a SAP group, then G satisfies the Kernel Unipotent property.

We can now prove the Kernel Unipotent property for G.

Proposition 4.10. — *The group G satisfies the Kernel Unipotent property.*

Proof. — From Lemma 4.4, we observe that for every positive integer n we have semi-direct product decompositions:

$$G/G_n \simeq F_{24}/F_{24,n} \rtimes_{\overline{\delta_F}} G_{13}/G_{13,n} \simeq F_{13}/F_{13,n} \rtimes_{\overline{\delta_{F'}}} G_{24}/G_{24,n}.$$

Let us consider $x := uh$ with $u \in F_{24}/F_{24,n}$ and $h \in G_{13}/G_{13,n}$, a nontrivial element in G/G_n. We need to construct a map $\rho_x \colon G/G_n \to \mathbb{U}_n$ such that $\rho_x(x) \neq \mathbb{I}_n$. For this purpose, we distinguish two cases:

(i) Assume that $h \neq 1$. By Lemma 4.8, the group G_{13} satisfies the Kernel Unipotent property, thus we have a map $\eta_h \colon G_{13}/G_{13,n} \to \mathbb{U}_n$ such that $\eta_h(h) \neq \mathbb{I}_n$. Let us define ρ_x by:

$$\rho_x|_{G_{13}/G_{13,n}} := \eta_h, \quad \text{and} \quad \rho_x|_{F_{24}/F_{24,n}} := \mathbb{1}.$$

This map is well-defined, and we have $\rho_x(x) := \rho_x(uh) = \eta_h(h) \neq \mathbb{I}_n$.

(ii) Now assume that $h = 1$. Then $x := u$ is in $F_{24}/F_{24,n}$, and not trivial. Additionally, Proposition 1.3 gives us the injection $F_{24}/F_{24,n} \subset G_{24}/G_{24,n}$. Since $G_{24} \simeq G_0$, we apply the same argument as (i), replacing G_{13} by G_{24} and F_{24} by F_{13}. □

Remark 4.11. — Expanding the previous proof, we can show that if G_1 and G_2 check the Kernel Unipotent property, then $G_1 \times G_2$ also satisfies it.

Observe that $F(2)$ also satisfies this property. We refer for instance to [37, Theorem 2.6]. Thus the group $G_\Gamma \simeq F_{13} \times F_{24}$ satisfies the Kernel Unipotent property. This group is not a maximal pro-2 quotient of an absolute Galois group. We refer to [45, Theorem 1.2].

4.4. The strong Massey Vanishing property. — In this subsection, we show that G satisfies the strong Massey Vanishing property. This subsection is heavily inspired by [43, §4.3] and we mostly use the same notations. Let us also recall that the structure of $H^\bullet(G)$ is given by Lemma 4.6 and Corollary 4.7.

Lemma 4.12. — *Assume that α and α' are nontrivial elements in $H^1(G)$ satisfying $\alpha \cup \alpha' = 0$. We have the following alternative. Either:*

- *α and α' are in $H^1(G_{13})$ and different from χ_0,*
- *α and α' are in $H^1(G_{24})$ and different from χ_0,*
- *α and α' are neither in $H^1(G_{13})$ nor in $H^1(G_{24})$, but they check the equality:*

$$\alpha' = \alpha + \chi_0.$$

Proof. — In this proof, we use the following fact coming from Corollary 4.7 and Lemma 4.6. If α is in $H^1(G_{13})$, then $\alpha \cup \psi_2 = \alpha \cup \psi_4 = 0$ if and only if $\alpha = 0$.

Let us take nontrivial elements α and α' in $H^1(G)$ such that $\alpha \cup \alpha' = 0$. We write

$$\alpha := a\chi_0 + \beta + b\psi_2 + c\psi_4, \quad \text{and} \quad \alpha' := a'\chi_0 + \beta' + b'\psi_2 + c'\psi_4,$$

with β, β' in the vector space generated by ψ_1 and ψ_3, that is identified with $H^1(F_{13}) \subset H^1(G_{13})$. Recall that we have the decomposition:

$$H^1(G_{13}) := H^1(F_{13}) \bigoplus \chi_0\mathbb{F}_2.$$

Using the relations from Lemma 4.6, we get the following expression for $\alpha \cup \alpha'$:

$$\begin{aligned}\alpha \cup \alpha' := aa'\chi_0^2 + a\chi_0\beta' + a'\beta\chi_0 + \beta\beta' + \\ (b\beta' + b'\beta + (ab' + a'b + bb')\chi_0) \cup \psi_2 + \\ (c\beta' + c'\beta + (ac' + a'c + cc')\chi_0) \cup \psi_4.\end{aligned}$$

Then, solving $\alpha \cup \alpha' = 0$, we identify the last two right-hand terms to infer the following system:

$$\begin{cases} b\beta' + b'\beta = 0 \\ ab' + a'b + bb' = 0 \\ c\beta' + c'\beta = 0, \\ ac' + a'c + cc' = 0. \end{cases}$$

We distinguish several cases:

(a) If we assume $(b, b') = (1, 0)$, then $\beta' = 0$ and $a' = 0$. So $c' = 1$. Thus either (i) $c = 0$ or (ii) $c = 1$. In the case (i), we have $\alpha' = \psi_4$, and $\alpha = \psi_2$. In the case (ii), we have $c = 1$. Thus $\beta = 0$ and $a = 1$ so $\alpha = \chi_0 + \psi_2 + \psi_4$, and $\alpha' = \psi_4$.

(b) The case $(b, b') = (0, 1)$ is symetric to the case (a). Consequently, in cases (a) and (b), we always infer that α and α' are in $H^1(G_{24})$.

(c) We assume that $(b, b') = (1, 1)$. This case imposes that $\beta = \beta'$ and $a' = a + 1$. We distinguish two cases. If (i) $\beta = 0$, then α and α' are both in $H^1(G_{24})$. If (ii) $\beta \neq 0$, then $c = c'$, so we infer that $\alpha' = \alpha + \chi_0$, and α, α' are neither in $H^1(G_{13})$ nor in $H^1(G_{24})$.

(d) We assume that $(b, b') = (0, 0)$. Since α and α' are nontrivial, we infer that $c = c'$. We have two case: either (i) $c = c' = 0$ or (ii) $c = c' = 1$. If (i) then α and α' are both in $H^1(G_{13})$. Else (ii) gives us $\beta = \beta'$ and $a = a' + 1$.

We distinguished all cases. To recap:

- in the cases (a), (b) and (c, i) we have α and α' both in $H^1(G_{24})$,
- in the case (d, i) we have α and α' both in $H^1(G_{13})$,
- in the cases (c, ii) and (d, ii), we have $\alpha' = \alpha + \chi_0$, and α and α' are neither in $H^1(G_{13})$ nor in $H^1(G_{24})$. □

Let us recall a result from Quadrelli [43, Lemma 4.2]:

Lemma 4.13. — *For $n > 2$, there exist matrices $A_1, A_2, B_1, B_2 \in \mathbb{U}_{n+1}$ such that: (i) the $(i, i+1)$-entries of both A_1 and A_2 are equal to 1, for $1 \leqslant i \leqslant n$, ($ii$) B_1 and B_2 are given by*

$$B_1 = \begin{pmatrix} 1 & 1 & & & & * \\ 0 & 1 & 0 & & & \\ & & 1 & 1 & & \\ & & & 1 & 0 & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 & & & & * \\ & 1 & 1 & & & \\ & & 1 & 0 & & \\ & & & 1 & 1 & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

and (iii) they satisfy $[B_1, A_1] = A_1^{-2}$ and $[B_2, A_2] = A_2^{-2}$.

We conclude this subsection.

Theorem 4.14. — *The group G satisfies the strong Massey Vanishing property.*

Proof. — Let us take a family $\{\alpha_1, \dots, \alpha_n\}$ of characters in $H^1(G)$ satisfying $\alpha_i \cup \alpha_{i+1} = 0$ for every $1 \leqslant i \leqslant n - 1$. We construct a morphism $\rho\colon G \to \mathbb{U}_{n+1}$ such that $\rho_{i,i+1} = \alpha_i$. From [44, Proposition 2.8], we can assume that $\alpha_i \neq 0$ for every i. Then from Lemma 4.12, we are in one of the following cases. Either:

(a) for every i, the character α_i is in $H^1(G_{13})$,

(b) for every i, the character α_i is in $H^1(G_{24})$,

(c) for every i, the character α_i is neither in $H^1(G_{13})$ nor in $H^1(G_{24})$, but satisfies the relation:

$$\alpha_{i+1} = \alpha_i + \chi_0.$$

We study the case (a). We observe that G_{13} is in $\mathscr{P}$. So G_{13} satisfies the strong Massey Vanishing property. Consequently, there exists a morphism $\eta_{13}\colon G_{13} \to \mathbb{U}_{n+1}$ such that $\eta_{13,i,i+1} = \alpha_i|_{G_{13}}$. Then we define:

$$\rho(x_1) = \eta_{13}(x_1), \quad \rho(x_3) = \eta_{13}(x_3), \quad \rho(x_0) = \eta_{13}(x_0), \quad \text{and } \rho(x_2) = \rho(x_4) = \mathbb{I}_{n+1}.$$

The case (b) is similar, since G_{24} is also in $\mathscr{P}$.

For the case (c), we infer that $\alpha_{2i} = \alpha_1 + \chi_0$, and $\alpha_{2i+1} = \alpha_1$. We consider A_1, A_2, B_1 and B_2 the matrices defined in Lemma 4.13. If $\alpha_1(x_0) = 1$, we take $A \coloneqq A_1$ and $B \coloneqq B_1$. Else we take $A \coloneqq A_2$ and $B \coloneqq B_2$.
Let us define $\rho(x_0) \coloneqq B$. Then we have

$$\rho(x_0)_{2i-1,2i} = \alpha_1(x_0), \quad \text{and } \rho(x_0)_{2i,2i+1} = \alpha_2(x_0).$$

If $1 \leqslant j \leqslant 4$, we observe that for every $1 \leqslant u, v \leqslant n$, we have $\alpha_u(x_j) = \alpha_v(x_j)$. Thus if $\alpha_1(x_j) = 0$, we define $\rho(x_j) \coloneqq \mathbb{I}_{n+1}$. If $\alpha_1(x_j) = 1$, we define $\rho(x_j) = A$. We obtain a morphism $\rho\colon G \to \mathbb{U}_{n+1}$ which satisifes $\rho_{i,i+1} = \alpha_i$.

□

4.5. Product of free groups. — Let us again consider Γ the square graph. The pro-2 group $G_\Gamma \simeq F_{13} \times F_{24}$ is already known not to be a maximal pro-2 quotient of an absolute Galois group. Quadrelli [42, Theorem 5.6] showed that this group is not Bloch-Kato, i.e. there exists a closed subgroup of G_Γ with non quadratic cohomology.

From Remark 4.11, the group G_Γ satisfies the Kernel Unipotent property. Furthermore, a presentation of G_Γ is given by:

$$\langle x_1, x_2, x_3, x_4 | \quad [x_1, x_2] = [x_2, x_3] = [x_3, x_4] = [x_1, x_4] = 1\rangle.$$

This presentation is minimal. In fact, it is mild. We refer to [16] for definitions. Furthermore the first author [11, Proposition 1.7] showed that $\mathscr{E}(G_\Gamma) \simeq \mathscr{E}_\Gamma$. Consequently, the group G_Γ satisfies the Koszul property. A presentation of $H^\bullet(G_\Gamma)$ is given by generators $\{\psi_1, \psi_2, \psi_3, \psi_4\}$ and relations:

$$\{\psi_i^2, \psi_1 \cup \psi_3, \psi_2 \cup \psi_4, \psi_u \cup \psi_v + \psi_v \cup \psi_u | \quad 1 \leqslant i \leqslant 4, 1 \leqslant u, v \leqslant 4\}.$$

In particular, a basis of $H^2(G_\Gamma)$ is given by $\{\psi_1 \cup \psi_2, \psi_2 \cup \psi_3, \psi_3 \cup \psi_4, \psi_1 \cup \psi_4\}$.

We recall that we denote by F_{13} and F_{24} the subgroups of G_Γ generated by $\{x_1, x_3\}$ and $\{x_2, x_4\}$. Let us also recall that these groups are pro-2 free on two generators. Thus they satisfy the strong Massey Vanishing property. Furthermore, we can identify $H^1(F_{13})$ with the vector space generated by $\{\psi_1, \psi_3\}$ and $H^1(F_{24})$ with the vector space generated by $\{\psi_2, \psi_4\}$. We also observe that $H^1(F_{13}) \cup H^1(F_{13}) = H^1(F_{24}) \cup H^1(F_{24}) = 0$. Furthermore, if we take α in $H^1(F_{13})$, we observe that $\alpha \cup \psi_2 = \alpha \cup \psi_4 = 0 \iff \alpha = 0$.

Similarly to Lemma 4.12, we have the following result.

Lemma 4.15. — *Let us take α and α' two nontrivial characters in $H^1(G_\Gamma)$ such that $\alpha \cup \alpha' = 0$. We have the following alternative. Either:*

(i) α and α' are in $H^1(F_{13})$,

(ii) or α and α' are in $H^1(F_{24})$,

(iii) or $\alpha = \alpha'$.

Proof. — The assertions (i) and (ii) are clear. Let us show (iii). We assume that α is neither in $H^1(F_{13})$ nor in $H^1(F_{24})$. Thus there exists β in $H^1(F_{13})$ different from zero and a couple $(a,b) \neq (0,0)$ in $\mathbb{F}_2^2$ such that $\alpha \coloneqq \beta + a\psi_2 + b\psi_3$.

Let us write $\alpha' \coloneqq \beta' + a'\psi_2 + b'\psi_4$ with $\beta' \in H^1(F_{13})$. Then from $\alpha \cup \alpha' = 0$, we infer:

$$\alpha \cup \alpha' = \beta \cup \beta' + (a\beta' + a'\beta) \cup \psi_2 + (b\beta' + b'\beta) \cup \psi_4 = 0.$$

Thus $a\beta' = a'\beta$ and $b\beta' = b'\beta$. Since $(a,b) \neq (0,0)$, we infer:

$$a = a', \quad b = b', \quad \beta = \beta', \quad \text{so} \quad \alpha = \alpha'.$$

□

Consequently, as Theorem 4.14, we can show that G_Γ checks the strong Massey Vanishing property.

Proposition 4.16. — *Let* Γ *be the square graph. Then the pro-*2 *RAAG* G_Γ *is not the maximal pro-*2 *quotient of an absolute Galois group, but satisfies the Koszul, the Kernel Unipotent and the strong Massey Vanishing property.*

Proof. — It remains to show that G_Γ checks the strong Massey Vanishing property. Let us consider a family $\alpha \coloneqq \{\alpha_1, \dots, \alpha_n\}$ of characters such that $\alpha_i \cup \alpha_{i+1} = 0$ for every $1 \leqslant i \leqslant n-1$. We construct a map $\rho\colon G_\Gamma \to \mathbb{U}_{n+1}$ such that $\rho_{i,i+1} = \alpha_i$ for $1 \leqslant i \leqslant n$.

If α_1 is either in $H^1(F_{13})$ or $H^1(F_{24})$, we conclude using the fact that F_{13} and F_{24} are free so check the strong Massey Vanishing property.

Assume that α_1 is neither in $H^1(F_{13})$ nor in $H^1(F_{24})$. Then for every $1 \leqslant i \leqslant n-1$, we have $\alpha_i = \alpha_{i+1}$. Let us define $A \coloneqq \mathbb{I}_{n+1} + \sum_{i=1}^{n} \delta_{i,i+1}$ where $\delta_{i,i+1}$ is the matrix which is zero everywhere except in $(i, i+1)$.

If $\alpha_1(x_i) = 1$, we define $\rho(x_i) \coloneqq A$. Else $\rho(x_i) \coloneqq \mathbb{I}_{n+1}$. Since A and $\mathbb{I}_{n+1}$ commutes, the morphism ρ is well-defined. Thus G_Γ checks the strong Massey Vanishing property. □

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